Geometric Sequences and Series

A geometric sequence is one in which there is a COMMON RATIO r between each pair of terms.

e.g.

u_1	u_2	u_3	u_4	....
3	6	12	24	

$$\frac{u_2}{u_1} = 2 \qquad \frac{u_3}{u_2} = 2 \qquad \frac{u_4}{u_3} = 2 \qquad \text{etc.}$$

The general geometric sequence is written

u_1	u_2	u_3	u_4	....	u_n
a	ar	ar^2	ar^3	-----	ar^{n-1}

Examples

① A savings account pays interest at 4% per year. The account is opened with £a.

(a) How much is in the account after 1 year?
after 2 years?
after n years?

(b) How long will it take for the amount to double?

(a) After 1 year, amount $= a \times 1.04$
2 years, $a \times 1.04^2$
n years, $a \times 1.04^n$

(b)
$$a \times 1.04^n = 2a$$
$(\div a)$
$$1.04^n = 2$$
$$\log_{10} 1.04^n = \log_{10} 2$$
$$n \log_{10} 1.04 = \log_{10} 2$$
$$n = \frac{\log_{10} 2}{\log_{10} 1.04} = \underline{17.7 \text{ years}}$$

alternative $\rightarrow$ $n = \log_{1.04} 2$
$= \underline{\underline{17.7 \text{ years}}}$

② The first and fifth terms of a geometric sequence are 3 and 12. Find the possible values of the 2nd, 3rd and 4th terms.

$$a = 3 \quad ①$$
$$ar^4 = 12 \quad ②$$

$\frac{②}{①}$
$$r^4 = 4$$
$$r^2 = 2$$
$$r = \sqrt{2} \text{ or } -\sqrt{2}$$

Second term $= ar = 3\sqrt{2}$ or $-3\sqrt{2}$
Third term $= ar^2 = 6$
Fourth term $= ar^3 = 6\sqrt{2}$ or $-6\sqrt{2}$

	u_1	u_2	u_3	u_4	u_5
$a=3,\ r=\sqrt{2}$	3	$3\sqrt{2}$	6	$6\sqrt{2}$	12
$a=3,\ r=-\sqrt{2}$	3	$-3\sqrt{2}$	6	$-6\sqrt{2}$	12

(−ve common ratio ⇒ terms alternative in sign)

A geometric series is the sum of a geometric sequence. The formula for the sum of n terms is found as follows.

$$S_n = a + ar + ar^2 + ar^3 + \cdots\cdots + ar^{n-1} \quad ①$$

(× both sides by r)

$$rS_n = ar + ar^2 + ar^3 + \cdots\cdots + ar^{n-1} + ar^n \quad ②$$

① − ②
$$S_n - rS_n = a - ar^n$$
$$S_n(1-r) = a(1-r^n)$$
$$\boxed{S_n = \frac{a(1-r^n)}{1-r}}$$
(in formula book)

Examples

③ Find $\sum_{r=0}^{7} (-2)^r$

This is $1 + (-2) + 4 + (-8) + \cdots\cdots + (-2)^7$

$a = 1 \quad r = -2 \quad n = 8$ (careful − from 0 to 7 is 8 terms)

$$\text{Sum} = \frac{1\left(1-(-2)^8\right)}{1-(-2)}$$

$$= \frac{1-256}{3} = \underline{\underline{-85}}$$

p 94 Ex 7A Q 1, 3
p 97 Ex 7B Q 1, 3, 4
p 99 Ex 7C Q 2, 3, 4, 5, 6, 8
p 102 Ex 7D Q 1 abcfh

Start now, finish by Monday

(4) Each year on 1st January, starting in 2001, Alex invests £1000 into an account paying 5% interest per year.

(a) How much is in the account by the end of 2010?

The first £1000 (from 2001) has grown to 1000×1.05^{10}

The second £1000 (from 2002) - - . 1000×1.05^{9}

:

The 10^{th} £1000 (from 2010) - . 1000×1.05

So total money =

$$(1000 \times 1.05) + (1000 \times 1.05^2) + (1000 \times 1.05^3) + \dots + (1000 \times 1.05^{10})$$

This is a geometric series with $r = 1.05$, $a = 1000 \times 1.05$ and $n = 10$

So

$$S_{10} = \frac{(1000 \times 1.05)(1-1.05^{10})}{1-1.05}$$

$$= \underline{\underline{£13206.79}}$$

(b) How many years will it take for the investment to be worth £20000?

$$\frac{(1000 \times 1.05)(1-1.05^n)}{1-1.05} = 20000$$

$$1050(1-1.05^n) = 20000 \times -0.05$$

$$1-1.05^n = -\frac{1000}{1050}$$

$$1 - 1.05^n = -0.95238$$
$$1.95238 = 1.05^n$$
$$\log_{1.05}(1.95238) = n$$
$$13.7 = n$$

The account will first exceed £20000 after 14 years

If $-1 < r < 1$, the terms of a geometric series get smaller and smaller, and the series has a <u>sum to infinity</u>. In the formula

$$S_n = \frac{a(1-r^n)}{1-r}$$

the expression r^n <u>tends to zero</u>, so we get

$$\boxed{S_\infty = \frac{a}{1-r}}$$

<u>Examples</u>

(5) Find the sum to infinity of

$$1 + \tfrac{1}{3} + \tfrac{1}{9} + \tfrac{1}{27} + \tfrac{1}{81} + \cdots\cdots$$

$$S_\infty = \frac{1}{1-\frac{1}{3}} = \frac{3}{2} \text{ or } 1\tfrac{1}{2}$$

(6) Find the sum to infinity of

$$1 - \tfrac{1}{3} + \tfrac{1}{9} - \tfrac{1}{27} + \tfrac{1}{81} \cdots\cdots$$

$$S_\infty = \frac{1}{1-(-\frac{1}{3})}$$

$$= \frac{1}{4/3} = \frac{3}{4}$$

(7) The second term of a geometric series is 6, and the sum to infinity is 8. Find the first term

$$ar = 2 \quad (1)$$

$$\frac{a}{1-r} = 8 \quad (2)$$

$$② \Rightarrow \quad a = 8(1-r)$$

$$\text{Subst in } ① \Rightarrow \quad 8(1-r)r = 2$$

$$8r - 8r^2 = 2$$

$$0 = 8r^2 - 8r + 2$$

$$0 = 4r^2 - 4r + 1$$

$$0 = (2r-1)(2r-1)$$

$$2r - 1 = 0 \Rightarrow r = \tfrac{1}{2}.$$

$$\text{Subst in } ① \Rightarrow \quad \tfrac{1}{2}a = 2$$

$$\underline{\underline{a = 4}}$$

p 103 Ex 7D Q 4, 5, 6, 8, 9
p 105 Ex 7E Q 1a-f, 2, 4, 7, 8