Geometric Series

Before we define what is meant by a series, we need to introduce a related topic, that of sequences. Formally, a **sequence** is a function that computes an ordered list. Suppose that on day 1, you have 1 dollar, and every day you double your money. Then the function $f(n) = 2^n$ generates the sequence

$$1, 2, 4, 8, 16, 32, \ldots,$$

when $n = 1, 2, 3, 4, 5, 6, \ldots$ This list represents the amount of dollars you have after n days. Note: The use of "…" is read as "and so on".

The individual entries in a sequence are called the **terms** of the sequence. In our discussion, we are going to assume that the terms in a particular sequence are real numbers.

Sequences can be grouped into two large classes based upon the number of terms they include. An **infinite sequence** is a function that has the set of natural numbers as its domain. As the name implies, it contains an infinite number of terms. In the opening example, the use of the "…" without some number on the end implies that the sequence continues indefinitely, following the prescribed pattern.

Of course, there is an inherent problem with assuming that money can be doubled forever. Instead, it makes sense to talk about doubling money for a certain number of days. Say, for $n = 1, 2, 3, 4, 5, 6$, and 7. In that case, the sequence generated would be called a **finite sequence**. Its domain is equal to a finite set of natural numbers. (In this case, $D = \{1, 2, \ldots, 7\}$.)

A common notation for sequences is let $a_n = f(n)$. With this notation, we say that a_n is the n^{th} term in the sequence.

Example 1:

Write out the first five terms a_1, a_2, a_3, a_4 and a_5 of the following sequences.

(a) $a_n = (-1)^n$

(b) $a_n = \sin\left(\frac{\pi}{2} n\right)$

(c) $a_n = 2n - 5$

(d) $a_n = 2(3)^{n-1}$

Solution:

(a) $a_1 = (-1)^1 = -1$, $a_2 = (-1)^2 = 1$, $a_3 = (-1)^3 = -1$, $a_4 = (-1)^4 = 1$, $a_5 = (-1)^5 = -1$.

(b) $a_1 = \sin\left(\frac{\pi}{2}(1)\right) = \sin\left(\frac{\pi}{2}\right) = 1$, $a_2 = \sin\left(\frac{\pi}{2}(2)\right) = \sin(\pi) = 0$,

$a_3 = \sin\left(\frac{\pi}{2}(3)\right) = \sin\left(\frac{3\pi}{2}\right) = -1$, $a_4 = \sin\left(\frac{\pi}{2}(4)\right) = \sin(2\pi) = 0$,

$a_5 = \sin\left(\frac{\pi}{2}(5)\right) = \sin\left(\frac{5\pi}{2}\right) = 1$.

(c) $a_1 = 2(1) - 5 = -3$, $a_2 = 2(2) - 5 = -1$, $a_3 = 2(3) - 5 = 1$, $a_4 = 2(4) - 5 = 3$,
$a_5 = 2(5) - 5 = 5$.

(d) $a_1 = 2(3)^{1-1} = 2(3)^0 = 2$, $a_2 = 2(3)^{2-1} = 2(3)^1 = 6$, $a_3 = 2(3)^{3-1} = 2(3)^2 = 18$,
$a_4 = 2(3)^{4-1} = 2(3)^3 = 54$, $a_5 = 2(3)^{5-1} = 2(3)^4 = 162$.

It is worth noting that using these formulas we would easily compute the 1,000th term in the sequence. We would only need to plug in $n = 1000$.

Some sequences are not written in terms of an explicit function like those above. Instead, they may be defined *recursively*, and hence are called a **recursive sequence**. That is, each term after the first few terms are defined in terms of what has come before it.

If it happens that the terms in our sequence are multiplies of each other (as was the case in Example 1d), then we say that we have a **geometric sequence**.

In Example 1d, the multiple between each term was 3. We call this number the **common ratio** and it is usually denoted by an r.

A geometric sequence can be defined recursively based on the common ratio between terms. That is, we have the relationship $a_n = ra_{n-1}$.

If we know the starting term of our sequence, a_1, since there is a common ratio r between subsequent terms, we can find an explicit formula for the n^{th} term of the sequence. Let us work out a few terms and try to discover the underlying pattern.

$a_2 = ra_1$
$a_3 = ra_2 = r(ra_1) = r^2a_1$
$a_4 = ra_3 = r(ra_2) = r(r(ra_1)) = r^3a_1$

In general, we have

The n^{th} term of a Geometric Sequence

In a geometric sequence with first term a_1 and common ratio r, the n^{th} term, a_n, is given by

$$a_n = a_1 r^{n-1}$$

Example 2:

Find a formula for the geometric sequence given by 2, 1, 1/2, 1/4, 1/8, …

Solution:

The first term is 2, so that is a_1. Notice that the common ratio between subsequent terms is 1/2 . So, we have that $r = 1/2$. Thus, $a_n = 2(1/2)^{n-1}$.

Example 3:

Find a general term a_n for the following geometric series if $a_2 = 4$ and $a_4 = 64$.

Solution:

We know that $a_2 = a_1 r$ and $a_4 = a_1 r^3$. So, if $4 = a_1 r$ and $64 = a_1 r^3$, we can divide the two equations to get $64/4 = (a_1 r^3)/(a_1 r) = r^2$, and we see that $r = 4$. Plugging that into the first equation, we $4 = a_1(4)$, so $a_1 = 1$. Thus, $a_n = 1(4)^{n-1} = 4^{n-1}$.

Now that we have established what is meant by a sequence and in particular a geometric series, we can turn our attention to a series.

Recall, a sequence is a function that computes an ordered list. A **series**, on the other hand, is the summation of elements generated by a sequence.

Let us return to our example with doubling money that we opened with. That is, suppose that on day 1, you have 1 dollar, and every day you double your money. Let's change the scenario slightly. Suppose on day 1 you have 1 dollar, but every day you are given twice the amount that you had the previous day. The function that specifies how much money you receive on the n^{th} days is given by $f(n) = 2^n$. This generates the sequence

$$1, 2, 4, 8, 16, 32, \ldots,$$

when $n = 1, 2, 3, 4, 5, 6, \ldots$ Suppose our interest is how much money you have after n days. (Remember, the above values are only how much you receive on a particular day.

You still get to keep your money from the previous days!) We would need to sum the values of our sequence up until day n to answer this question.

As was the case with sequence, series can be grouped into two large classes based upon the number of terms they include. An **infinite series** is the summation of the terms in an infinite sequence. A **finite series** is the summation of the terms in a finite sequence. We shall consider both types of series.

We define a **geometric series** as the summation of the terms in a geometric sequence.

We can use the formula for the n^{th} term of the geometric sequence to develop a formula for the sum of the first n terms in a geometric sequence.

Recall, if a_1 was the first term in the geometric sequence with a common ratio of r, then the formula for the n^{th} term in a geometric sequence is given by $a_n = a_1 r^{n-1}$.

Let S_n denote the sum of the first n terms in a geometric sequence. Then we have:

$$S_n = a_1 + a_1 r + a_1 r^2 + \ldots + a_1 r^{n-2} + a_1 r^{n-1}$$

Multiplying this equation by r, we have

$$rS_n = a_1 r + a_1 r^2 + \ldots + a_1 r^{n-1} + a_1 r^n$$

Subtracting this equation from S_n, we have

$$\begin{aligned} S_n - rS_n &= (a_1 + a_1 r + a_1 r^2 + \ldots + a_1 r^{n-2} + a_1 r^{n-1}) - (a_1 r + a_1 r^2 + \ldots + a_1 r^{n-1} + a_1 r^n) \\ &= a_1 - a_1 r^n, \end{aligned}$$

since all of the middle terms cancelled out. We can factor a S_n out of the terms on the left-hand side and a a_1 out of the terms on the right-hand side to get $S_n(1 - r) = a_1(1 - r^n)$.

And so, we have that

Sum of the first *n* terms of a Geometric Sequence

If a geometric sequence has first term a_1 and common ratio r, then the sum of the first n terms is given by

$$S_n = a_1\left(\frac{1-r^n}{1-r}\right), \quad \text{provided } r \neq 1$$

Notice that we need to make the assumption that $r \neq 1$, since we divided both sides by $1 - r$, would be 0 if $r = 1$. If it were the case that $r = 1$, then our geometric series actually reduces to an arithmetic series with $d = 0$.

Example 4:

Find the sum for the given values of *n*.

$3 - 6 + 12 - 24 + 48 - \dots + 3(-2)^{n-1}$; $n = 4, 7$, and 10.

Solution:

This is a geometric series with first term $a_1 = 3$ and common ratio $r = -2$.

$$S_4 = 3\left(\frac{1-(-2)^4}{1-(-2)}\right) = -15, \quad S_7 = 3\left(\frac{1-(-2)^7}{1-(-2)}\right) = 129, \quad S_{10} = 3\left(\frac{1-(-2)^{10}}{1-(-2)}\right) = -1023$$

Example 5:

Find the following sum: 1 + 2 + 4 + 8 + 16 + 32 + 64 + 128

Solution:

This is a geometric series with the first term $a_1 = 1$ and common ratio $r = 2$. We are adding up the first 8 terms. Thus, we have $S_8 = 1\left(\frac{1-(2)^8}{1-2}\right) = 255$.

So far, we have restricted our attention to finite series. There are some infinite geometric series for which the sum is a finite number. The ancient Greek Zeno first proposed a variant of the following problem.

Suppose a person wants to walk through a forest that is one mile wide. Suppose he walks half the distance in an hour. Then in the next hour, walks half of the remaining distance, and continues in this manner. How far will the person have walked? How long will it take the person to leave the forest?

The distance traveled by the person is described by an infinite series. Namely,

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$$

Intuition tells us that the person will walk 1 mile (the total width of the forest). But since the person walks slower and slower, it will take an infinite amount of time to travel that distance. So, the person never actually leaves the forest!

Looking at our formula for the finite geometric series, notice that if $|r| < 1$, then as n gets large, r^n approaches 0. That is, if $|r| < 1$, then $\lim_{n\to\infty} r^n = 0$. Thus, we have

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} a_1\left(\frac{1-r^n}{1-r}\right) = a_1\left(\frac{1-0}{1-r}\right) = \frac{a_1}{1-r}.$$

This is summarized as

Sum of an infinite Geometric Sequence

The sum of the infinite geometric sequence with first term a_1 and common ratio r is given by

$$S = \frac{a_1}{1-r}, \quad \text{provided } |r| < 1.$$

If $|r| \geq 1$, then the sum either does not exist or is infinite.

Example 6:

Show that the sum of the infinite geometric sequence $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+...$ equals 1.

Solution:

The first term in the sequence is $a_1 = 1/2$ and the common ratio is $r = 1/2$. And since $|1/2| < 1$, we can use the formula above to conclude that $S = \frac{\frac{1}{2}}{1-\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1$.

We can use infinite series to expression fractions as summations. This is done by rewriting the fraction with a denominator of 1 – 0.1, so our common ratio will be 0.1, which corresponds to one decimal place.

Example 7:

Write 2/3 as an infinite geometric series.

Solution:

Observe that $\frac{2}{3} = \frac{6}{9} = \frac{0.6}{1-0.1}$. This is in the form an infinite geometric series with a_1 = 0.6 and $r = 0.1$. Thus, we have that $\frac{2}{3} = 0.6 + 0.06 + 0.006 + ...$