12.3 ARITHMETIC SEQUENCES AND SERIES

We defined sequences and series in Sections 12.1 and 12.2. In this section you will study a special type of sequence known as an arithmetic sequence. You will also study the series corresponding to this sequence.

Arithmetic Sequences

Consider the following sequence:

$$5, 9, 13, 17, 21, \ldots$$

This sequence is called an arithmetic sequence because of the pattern for the terms. Each term is 4 larger than the previous term.

> **Arithmetic Sequence**
>
> A sequence in which each term after the first is obtained by adding a fixed amount to the previous term is called an **arithmetic sequence.**

The fixed amount is called the **common difference** and is denoted by the letter d. If a_1 is the first term, then the second term is $a_1 + d$. The third term is $a_1 + 2d$, the fourth term is $a_1 + 3d$, and so on.

> **Formula for the nth Term of an Arithmetic Sequence**
>
> The nth term, a_n, of an arithmetic sequence with first term a_1 and common difference d is
>
> $$a_n = a_1 + (n - 1)d.$$

EXAMPLE 1   **The nth term of an arithmetic sequence**

Write a formula for the nth term of the arithmetic sequence

$$5, 9, 13, 17, 21, \ldots.$$

Solution

Each term of the sequence after the first is 4 more than the previous term. Because the common difference is 4 and the first term is 5, the nth term is given by

$$a_n = 5 + (n - 1)4.$$

We can simplify this expression to get

$$a_n = 4n + 1.$$ ∎

In the next example the common difference is negative.

EXAMPLE 2   **An arithmetic sequence of decreasing terms**

Write a formula for the nth term of the arithmetic sequence

$$4, 1, -2, -5, -8, \ldots.$$

Solution

Each term is 3 less than the previous term, so $d = -3$. Because $a_1 = 4$, we can write the nth term as

$$a_n = 4 + (n - 1)(-3),$$

or

$$a_n = -3n + 7. \qquad \blacksquare$$

In the next example we find some terms of an arithmetic sequence using a given formula for the nth term.

E X A M P L E   3

Writing terms of an arithmetic sequence

Write the first five terms of the sequence in which $a_n = 3 + (n - 1)6$.

Solution

Let n take the values from 1 through 5, and find a_n:

$$a_1 = 3 + (1 - 1)6 = 3$$
$$a_2 = 3 + (2 - 1)6 = 9$$
$$a_3 = 3 + (3 - 1)6 = 15$$
$$a_4 = 3 + (4 - 1)6 = 21$$
$$a_5 = 3 + (5 - 1)6 = 27$$

Notice that $a_n = 3 + (n - 1)6$ gives the general term for an arithmetic sequence with first term 3 and common difference 6. Because each term after the first is 6 more than the previous term, the first five terms that we found are correct. $\qquad \blacksquare$

The formula $a_n = a_1 + (n - 1)d$ involves four variables: a_1, a_n, n, and d. If we know the values of any three of these variables, we can find the fourth.

E X A M P L E   4

Finding a missing term of an arithmetic sequence

Find the twelfth term of the arithmetic sequence whose first term is 2 and whose fifth term is 14.

Solution

Before finding the twelfth term, we use the given information to find the missing common difference. Let $n = 5$, $a_1 = 2$, and $a_5 = 14$ in the formula $a_n = a_1 + (n - 1)d$ to find d:

$$14 = 2 + (5 - 1)d$$
$$14 = 2 + 4d$$
$$12 = 4d$$
$$3 = d$$

Now use $a_1 = 2$, $d = 3$ and $n = 12$ in $a_n = a_1 + (n - 1)d$ to find a_{12}:

$$a_{12} = 2 + (12 - 1)3$$
$$a_{12} = 35 \qquad \blacksquare$$

study tip

Stay alert for the entire class period. The first 20 minutes are the easiest and the last 20 minutes are the hardest. Some students put down their pencils, fold up their notebooks, and daydream for those last 20 minutes. Don't give in. Recognize when you are losing it and force yourself to stay alert. Think of how much time you will have to spend outside of class figuring out what happened during those last 20 minutes.

Arithmetic Series

The indicated sum of an arithmetic sequence is called an **arithmetic series.** For example, the series

$$2 + 4 + 6 + 8 + 10 + \cdots + 54$$

is an arithmetic series because there is a common difference of 2 between the terms.

We can find the actual sum of this arithmetic series without adding all of the terms. Write the series in increasing order, and below that write the series in decreasing order. We then add the corresponding terms:

$$
\begin{aligned}
S &= 2 + 4 + 6 + 8 + \cdots + 52 + 54 \\
S &= 54 + 52 + 50 + 48 + \cdots + 4 + 2 \\
\hline
2S &= 56 + 56 + 56 + 56 + \cdots + 56 + 56
\end{aligned}
$$

Now, how many times does 56 appear in the sum on the right? Because

$$2 + 4 + 6 + \cdots + 54 = 2 \cdot 1 + 2 \cdot 2 + 2 \cdot 3 + \cdots + 2 \cdot 27,$$

there are 27 terms in this sum. Because 56 appears 27 times on the right, we have $2S = 27 \cdot 56$, or

$$S = \frac{27 \cdot 56}{2} = 27 \cdot 28 = 756.$$

If $S_n = a_1 + a_2 + a_3 + \cdots + a_n$ is any arithmetic series, then we can find its sum using the same technique. Rewrite S_n as follows:

$$
\begin{aligned}
S_n &= a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + a_n \\
S_n &= a_n + (a_n - d) + (a_n - 2d) + \cdots + a_1 \\
\hline
2S_n &= (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) + \cdots + (a_1 + a_n) \quad \text{Add.}
\end{aligned}
$$

Because $(a_1 + a_n)$ appears n times on the right, we have $2S_n = n(a_1 + a_n)$. Divide each side by 2 to get the following formula.

> **Sum of an Arithmetic Series**
>
> The sum, S_n, of the first n terms of an arithmetic series with first term a_1 and nth term a_n, is given by
>
> $$S_n = \frac{n}{2}(a_1 + a_n).$$

E X A M P L E   5

The sum of an arithmetic series

Find the sum of the positive integers from 1 to 100 inclusive.

Solution

The described series, $1 + 2 + 3 + \cdots + 100$, has 100 terms. So we can use $n = 100$, $a_1 = 1$, and $a_n = 100$ in the formula for the sum of an arithmetic series:

$$S_n = \frac{n}{2}(a_1 + a_n)$$

$$S_{100} = \frac{100}{2}(1 + 100)$$

$$= 50(101)$$

$$= 5050$$

helpful hint

Legend has it that Carl F. Gauss knew this formula when he was in grade school. Gauss's teacher told him to add up the numbers from 1 through 100 for busy work. He immediately answered 5050.

E X A M P L E   6

The sum of an arithmetic series

Find the sum of the series

$$12 + 16 + 20 + \cdots + 84.$$

Solution

This series is an arithmetic series with $a_n = 84$, $a_1 = 12$, and $d = 4$. To get the number of terms, n, we use $a_n = a_1 + (n - 1)d$:

$$84 = 12 + (n - 1)4$$
$$84 = 8 + 4n$$
$$76 = 4n$$
$$19 = n$$

Now find the sum of these 19 terms:

$$S_{19} = \frac{19}{2}(12 + 84) = 912$$

■

WARM-UPS

True or false? Explain your answer.

1. The arithmetic sequence 3, 1, −1, −3, −5, . . . has common difference 2.

2. The sequence 2, 5, 9, 14, 20, 27, . . . is an arithmetic sequence.

3. The sequence 2, 4, 2, 0, 2, 4, 2, 0, . . . is an arithmetic sequence.

4. The nth term of an arithmetic sequence with first term a_1 and common difference d is given by the formula $a_n = a_1 + nd$.

5. If $a_1 = 5$ and $a_3 = 10$ in an arithmetic sequence, then $a_4 = 15$.

6. If $a_1 = 6$ and $a_3 = 2$ in an arithmetic sequence, then $a_2 = 10$.

7. An arithmetic series is the indicated sum of an arithmetic sequence.

8. The series $\sum_{i=1}^{5} (3 + 2i)$ is an arithmetic series.

9. The sum of the first n counting numbers is $\frac{n(n + 1)}{2}$.

10. The sum of the even integers from 8 through 28 inclusive is $5(8 + 28)$.

12.3   EXERCISES

Reading and Writing   *After reading this section, write out the answers to these questions. Use complete sentences.*

1. What is an arithmetic sequence?

2. What is the nth term of an arithmetic sequence?

3. What is an arithmetic series?

4. What is the formula for the sum of the first n terms of an arithmetic series?

Write a formula for the nth term of each arithmetic sequence. See Examples 1 and 2.

5. 0, 6, 12, 18, 24, . . .

6. 0, 5, 10, 15, 20, . . .

7. 7, 12, 17, 22, 27, . . .

8. 4, 15, 26, 37, 48, . . .

9. −4, −2, 0, 2, 4, . . .

10. $-3, 0, 3, 6, 9, \ldots$

11. $5, 1, -3, -7, -11, \ldots$

12. $8, 5, 2, -1, -4, \ldots$

13. $-2, -9, -16, -23, \ldots$

14. $-5, -7, -9, -11, -13, \ldots$

15. $-3, -2.5, -2, -1.5, -1, \ldots$

16. $-2, -1.25, -0.5, 0.25, \ldots$

17. $-6, -6.5, -7, -7.5, -8, \ldots$

18. $1, 0.5, 0, -0.5, -1, \ldots$

In Exercises 19–32, write the first five terms of the arithmetic sequence whose nth term is given. See Example 3.

19. $a_n = 9 + (n-1)4$

20. $a_n = 13 + (n-1)6$

21. $a_n = 7 + (n-1)(-2)$

22. $a_n = 6 + (n-1)(-3)$

23. $a_n = -4 + (n-1)3$

24. $a_n = -19 + (n-1)12$

25. $a_n = -2 + (n-1)(-3)$

26. $a_n = -1 + (n-1)(-2)$

27. $a_n = -4n - 3$

28. $a_n = -3n + 1$

29. $a_n = 0.5n + 4$

30. $a_n = 0.3n + 1$

31. $a_n = 20n + 1000$

32. $a_n = -600n + 4000$

Find the indicated part of each arithmetic sequence. See Example 4.

33. Find the eighth term of the sequence that has a first term of 9 and a common difference of 6.

34. Find the twelfth term of the sequence that has a first term of -2 and a common difference of -3.

35. Find the common difference if the first term is 6 and the twentieth term is 82.

36. Find the common difference if the first term is -8 and the ninth term is -64.

37. If the common difference is -2 and the seventh term is 14, then what is the first term?

38. If the common difference is 5 and the twelfth term is -7, then what is the first term?

39. Find the sixth term of the sequence that has a fifth term of 13 and a first term of -3.

40. Find the eighth term of the sequence that has a sixth term of -42 and a first term of 3.

Find the sum of each given series. See Examples 5 and 6.

41. $1 + 2 + 3 + \cdots + 48$

42. $1 + 2 + 3 + \cdots + 12$

43. $8 + 10 + 12 + \cdots + 36$

44. $9 + 12 + 15 + \cdots + 72$

45. $-1 + (-7) + (-13) + \cdots + (-73)$

46. $-7 + (-12) + (-17) + \cdots + (-72)$

47. $-6 + (-1) + 4 + 9 + \cdots + 64$

48. $-9 + (-1) + 7 + \cdots + 103$

49. $20 + 12 + 4 + (-4) + \cdots + (-92)$

50. $19 + 1 + (-17) + \cdots + (-125)$

51. $\displaystyle\sum_{i=1}^{12} (3i - 7)$

52. $\displaystyle\sum_{i=1}^{7} (-4i + 6)$

53. $\displaystyle\sum_{i=1}^{11} (-5i + 2)$

54. $\displaystyle\sum_{i=1}^{19} (3i - 5)$

Solve each problem using the ideas of arithmetic sequences and series.

55. **Increasing salary.** If a lab technician has a salary of $22,000 her first year and is due to get a $500 raise each year, then what will her salary be in her seventh year?
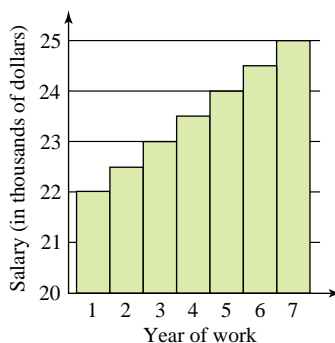

FIGURE FOR EXERCISE 55

56. **Seven years of salary.** What is the total salary for 7 years of work for the lab technician of Exercise 55?

57. **Light reading.** On the first day of October an English teacher suggests to his students that they read five pages of

a novel and every day thereafter increase their daily reading by two pages. If his students follow this suggestion, then how many pages will they read during October?

58. *Heavy penalties.* If an air-conditioning system is not completed by the agreed upon date, the contractor pays a penalty of $500 for the first day that it is overdue, $600 for the second day, $700 for the third day, and so on. If the system is completed 10 days late, then what is the total amount of the penalties that the contractor must pay?

GETTING MORE INVOLVED

59. *Discussion.* Which of the following sequences is not an arithmetic sequence? Explain your answer.

a) $\dfrac{1}{2}, 1, \dfrac{3}{2}, \ldots$ b) $\dfrac{1}{2}, \dfrac{1}{3}, \dfrac{1}{4}, \ldots$

c) $5, 0, -5, \ldots$ d) $2, 3, 4, \ldots$

60. *Discussion.* What is the smallest value of n for which

$$\sum_{i=1}^{n} \frac{i}{2} > 50?$$

12.4 **GEOMETRIC SEQUENCES AND SERIES**

In this section

- Geometric Sequences
- Finite Geometric Series
- Infinite Geometric Series
- Annuities

In Section 12.3 you studied the arithmetic sequences and series. In this section you will study sequences in which each term is a *multiple* of the term preceding it. You will also learn how to find the sum of the corresponding series.

Geometric Sequences

Consider the following sequence:

$$3, 6, 12, 24, 48, \ldots$$

Unlike an arithmetic sequence, these terms do not have a common difference, but there is a simple pattern to the terms. Each term after the first is twice the term preceding it. Such a sequence is called a geometric sequence.

> **Geometric Sequence**
>
> A sequence in which each term after the first is obtained by multiplying the preceding term by a constant is called a **geometric sequence.**

The constant is denoted by the letter r and is called the **common ratio.** If a_1 is the first term, then the second term is $a_1 r$. The third term is $a_1 r^2$, the fourth term is $a_1 r^3$, and so on. We can write a formula for the nth term of a geometric sequence by following this pattern.

> **Formula for the *n*th Term of a Geometric Sequence**
>
> The nth term, a_n, of a geometric sequence with first term a_1 and common ratio r is
>
> $$a_n = a_1 r^{n-1}.$$

The first term and the common ratio determine all of the terms of a geometric sequence.