6 EXTENDING ALGEBRA

Objectives

After studying this chapter you should

- understand techniques whereby equations of cubic degree and higher can be solved;
- be able to factorise polynomials;
- be able to use the remainder theorem.

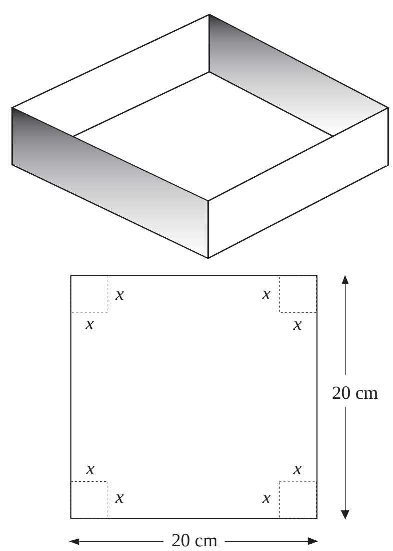
6.0 Introduction

This chapter is different from the others in this book; although it begins with a realistic example, much of the rest does not have any immediate relevance to the outside world. Its purpose is to extend the algebra you already know.

This does not make it a dull chapter. On the contrary, there are many people who enjoy abstract mathematics as being interesting in its own right. Over the centuries mathematical techniques have developed partly in response to problems that needed to be solved, but there have always been those who have pondered on the less applicable side of the subject purely out of interest and curiosity.

6.1 The cubic equation

How can an open-topped box with volume 500 cm^3 be made from a square piece of card 20 cm by 20 cm?



To make such a box, a net can be made by cutting off equal squares from each corner, as shown in the diagram.

Suppose each smaller square has side x cm. The dimensions of the box will then be x by $(20-2x)$ by $(20-2x)$. For the volume to be 500 cm^3, x must satisfy the equation

$$x(20-2x)^2 = 500, \quad 0 \le x \le 10$$

When the expression has been multiplied out, the highest power of x in this equation is x^3; accordingly this equation is called a **cubic equation**.

As with a quadratic equation, it is good practice to reduce this equation to the form $f(x) = 0$.

$$x(400 - 80x + 4x^2) = 500$$

$$\Rightarrow \quad 4x^3 - 80x^2 + 400x - 500 = 0$$

$$\Rightarrow \quad x^3 - 20x^2 + 100x - 125 = 0, \ \ 0 \le x \le 10$$

Again by comparison with quadratic equations, the next thing you might want to do would be to try to factorise it. With a cubic function, however, this is not easy.

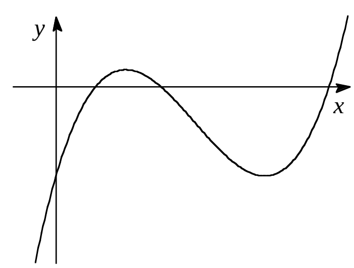
Fortunately one solution to the problem can be seen 'by inspection'. A little experimentation reveals that $x = 5$ will fit the bill. Is this the only solution? Remember that for quadratic equations where there was one answer there was almost always another.

One method of solution is to draw the graph of the cubic function $x^3 - 20x^2 + 100x - 125$ and see where it crosses the x-axis.

Activity 1 Using a graph

Sketch the graph of the function

$$f(x) = x^3 - 20x^2 + 100x - 125$$



Verify that $f(5) = 0$ and find any other values of x where the curve crosses the horizontal axis.

Bearing in mind the domain of x, solve the original problem to the nearest millimetre.

The solutions of an equation are called the **roots** of an equation.

Hence 5 is a root of $x^3 - 20x^2 + 100x - 125 = 0$ and Activity 1 should have revealed that there are two other roots as well. Another way of saying the same thing is to refer to 5 and the two other solutions as being **zeros** of the function; in other words, these values make that function equal to zero.

$$\boxed{\alpha \text{ is a root of } f(x) = 0 \iff f(\alpha) = 0}$$

How could the cubic equation have been solved without the need to draw the graph? Activity 2 gives a clue.

Activity 2 Zeros and factors

You may find it useful to use a graphic calculator.

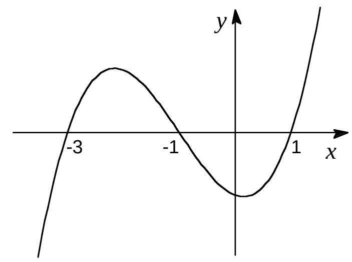
(a) Sketch the graph of the function $x^2 + 2x - 143$. What is the factorised form of this function? How can you tell by looking at the graph?

(b) Now sketch the graph of the cubic function $(x+4)(x-1)(x-3)$. How does the form of the function correspond to its zeros?

(c) Sketch the graph of the function $x^3 - 79x + 210$. Hence write this function as the product of three linear factors.

(d) Can you factorise $x^3 - 5x^2 + 3x + 4$ by looking at its graph?

(e) Suggest a possible equation for the curve on the right.



Now look again at the equation

$$x^3 - 20x^2 + 100x - 125 = 0$$

$x = 5$ is known to be a zero of the function $x^3 - 20x^2 + 100x - 125$ and so it can be deduced, from your work on Activity 2 that $(x-5)$ is a factor.

If $(x-5)$ is a factor, how does the rest of the factorisation go? It is evident that $x^3 - 20x^2 + 100x - 125 = (x-5) \times$ (quadratic function). Moreover, the quadratic must be of the form

$$x^2 + bx + 25$$

Finding b is not quite so obvious but can be done as follows :

$$x^3 - 20x^2 + 100x - 125 = (x-5)(x^2 + bx + 25)$$

To make $-20x^2$ in the cubic equation:

$$-20 = -5 + b \;\Rightarrow b = -15$$

To make $+100x$ in the cubic equation:

$$100 = 25 - 5b \;\Rightarrow b = -15$$

Hence $x^3 - 20x^2 + 100x - 125 = (x-5)(x^2 - 15x + 25)$

The solution to the cubic equation is as follows

$$x^3 - 20x^2 + 100x - 125 = 0$$

$$\Rightarrow \quad (x-5)(x^2 - 15x + 25) = 0$$

$$\Rightarrow \quad \text{either } x - 5 = 0 \text{ or } \quad x^2 - 15x + 25 = 0$$

$$\Rightarrow \quad x = 1.9,\ 5 \text{ or } 13.1 \text{ to } 1 \text{ d.p.}$$

Example

Find all solutions of $x^3 - 3x^2 - 33x + 35 = 0$

Solution

A bit of searching reveals that $x = 1$ is a root of this equation, since $f(1) = 0$. Hence $(x-1)$ must be a factor, and you can write

$$x^3 - 3x^2 - 33x + 35 = (x-1)(x^2 + bx - 35)$$

To find b : the cubic equation contains $-33x$, so

$$-33 = -b - 35$$

$$\Rightarrow \quad b = -2$$

$$\Rightarrow \quad x^3 - 3x^2 - 33x + 35 = (x-1)(x^2 - 2x - 35)$$

In this case, the quadratic factor itself factorises to

$$(x-7)(x+5)$$

so

$$x^3 - 3x^2 - 33x + 35 = (x-1)(x-7)(x+5)$$

The original equation can thus be re-written

$$(x-1)(x-7)(x+5) = 0$$

$$\Rightarrow \quad x = -5, 1 \text{ or } 7$$

Example

Solve $x^3 - 14x - 15 = 0$

Solution

$x = -3$ is a root of this equation. Hence $(x+3)$ is a factor.

$$x^3 - 14x - 15 = (x+3)(x^2 + bx - 5)$$

To find b: cubic equation has $0x^2$, so

$$0 = b + 3$$

$$\Rightarrow \quad b = -3$$

$$\Rightarrow \quad x^3 - 14x - 15 = (x+3)(x^2 - 3x - 5)$$

The original equation thus reads

$$(x+3)(x^2 - 3x - 5) = 0$$

$$\Rightarrow \quad \text{either } x + 3 = 0 \text{ or } x^2 - 3x - 5 = 0$$

$$\Rightarrow \quad x = -3, -1.19, \text{ or } 4.19, \text{ to } 3 \text{ s.f.}$$

Activity 3 Missing roots

(a) Find a linear factor of $x^3 - 4x^2 - 2x + 20$ and the corresponding quadratic factor. Hence find all the solutions of $x^3 - 4x^2 - 2x + 20 = 0$.

(b) Illustrate your answer by means of a sketch graph.

Exercise 6A

1. (a) Work out the missing quadratic factors

   (i) $x^3 - 3x^2 - 6x + 8 = (x-4)(\quad)$

   (ii) $x^3 + 8x^2 + 12x - 9 = (x+3)(\quad)$

   (iii) $2x^3 - x^2 - 117x - 324 = (2x+9)(\quad)$

   (b) Use your answers to (a) to find all the roots of

   (i) $x^3 - 3x^2 - 6x + 8 = 0$

   (ii) $x^3 + 8x^2 + 12x - 9 = 0$

   (iii) $2x^3 - x^2 - 117x - 324 = 0$

2. Explain how you know that

   (a) $(x-3)$ is a factor of $x^3 - 2x^2 + x - 12$

   (b) $(x+5)$ is a factor of $2x^3 + 6x^2 - 23x - 15$

   (c) $(2x-1)$ is a factor of $4x^3 + 2x^2 + 8x - 5$.

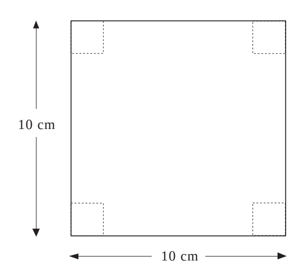
3. Find all the roots of these equations

   (a) $x^3 - 5x^2 + 6x - 2 = 0$

   (b) $x^3 + 3x^2 - 46x = 48$

   (c) $2x^3 - x^2 - 18x + 9 = 0$

4. Four identical 'square corners' are cut from a square piece of card measuring 10 cm by 10 cm. The resulting net will make an open topped box with volume 64 cm^3. Find the size of the squares that must be removed.



6.2 No simple solution

You should by now have asked the question : "What happens if no simple solution can be found?" The process you have used depends crucially on first being able to find a root. You may wonder whether, as with quadratic equations that cannot be factorised, there is a formula that will give all the roots automatically.

The answer is that a systematic method of finding roots of cubics does exist. The bad news, however, is that

(a) it is a long-winded method which is seldom used;

(b) it involves complicated maths.

A practical solution is to use either a graphical method or trial and improvement. Both methods can be time-consuming and both depend on first knowing approximately where the roots are.

For example, to solve the equation $x^3 - 100x^2 + 2000x - 1500 = 0$ it is a help to work out a few values of the function first. Let the function be labelled $f(x)$.

A table of values is shown on the right. The arrows show where $f(x)$ changes from positive to negative or vice versa. Hence the zeros of $f(x)$ must occur

between 0 and 10

between 20 and 30

between 70 and 80

Trial and improvement, or graphs drawn in the correct regions, eventually give answers of 0.78, 26.40 and 72.82 to 2 d.p.

x	$f(x)$	
−30	−178500	
−20	−89500	
−10	−32500	
0	−1500	←
10	9500	
20	6500	←
30	−4500	
40	−17500	
50	−26500	
60	−25500	
70	−8500	←
80	30500	

Exercise 6B

1. (a) Explain how you can tell that the function $f(x) = x^3 + x^2 - x + 5$ has a zero between $x = -3$ and $x = -2$.

   (b) Find this value to 3 s.f.

2. Solve these cubic equations to 3 s.f.

   (a) $2x^3 - 150x^2 + 75000 = 0$

   (b) $x^3 + 4x^2 - 32x - 100 = 0$

6.3 Factor theorem

Quadratics and cubics are particular examples of **polynomial functions** : a quadratic function is a polynomial of degree 2 : a cubic has degree 3. In general a **polynomial of degree** *n* has the form

$$a_n x^n + a_{n-1} x^{n-1} + \ \ldots \ + a_2 x^2 + a_1 x + a_0$$

where $a_0, \ a_1, \ \ldots \ a_n$ are real numbers, and $a_n \neq 0$. When solving cubic equations, the spotting of a root leads immediately to a simple linear factor : e.g.

$$\begin{aligned} & 3 \text{ is a zero of } x^3 - 5x^2 + 11x - 15 \\ \Rightarrow \quad & (x-3) \text{ is a factor of } x^3 - 5x^2 + 11x - 15 \\ \Rightarrow \quad & x^3 + 5x^2 + 11x - 15 = (x-3) \times (\text{quadratic}) \end{aligned}$$

The same technique can be applied to polynomials of any degree. For example,

$$\begin{aligned} & 2 \text{ is a zero of } x^4 + 3x^2 - 17x + 6 \\ \Rightarrow \quad & (x-2) \text{ is a factor of } x^4 + 3x^2 - 17x + 6 \\ \Rightarrow \quad & x^4 + 3x^2 - 17x + 6 = (x-2) \times (\text{cubic}) \end{aligned}$$

In general

> If $P(x)$ is a polynomial of degree *n* and has a zero at $x = \alpha$, that is $P(\alpha) = 0$, then $(x - \alpha)$ is a factor of $P(x)$ and
>
> $$P(x) = (x - \alpha) Q(x)$$
>
> where $Q(x)$ is a polynomial of degree $(n-1)$

This is known as the **factor theorem.**

Example

Show that $(x-5)$ is a factor of the polynomial

$$x^5 - 4x^4 - x^3 - 21x^2 + 25.$$

Solution

Denote the polynomial by $P(x)$.

$$\begin{aligned} P(5) &= 5^5 - 4\times 5^4 - 5^3 - 21\times 5^2 + 25 \\ &= 3125 - 2500 - 125 - 525 + 25 \\ &= 0 \end{aligned}$$

Hence 5 is a zero of $P(x)$ and so by the factor theorem, $(x-5)$ is a factor.

Activity 4 Fractions as roots

(a) Show that $x = \frac{1}{2}$ is a zero of $2x^3 - 3x^2 - 3x + 2$.

(b) $2x^3 - 3x^2 - 3x + 2$ has two linear factors other than $x = \frac{1}{2}$. Find them. Check that all three factors multiply together to give the original cubic.

(c) Find the quadratic expression missing from this statement :

$$2x^3 - 3x^2 - 3x + 2 = (2x-1)(\qquad)$$

(d) Suppose the fraction $\dfrac{p}{q}$ is a root of a polynomial. What could the associated factor be?

Activity 5 Tips for root-spotting

(a) Just by looking at the equation $x^3 + 3x^2 + 3x + 2 = 0$ it is possible to deduce that, if there is an integer root, it can only be ± 1 or ± 2. Why? What is the root?

(b) If there is an integer root of $x^4 + 3x^3 + x^2 + 2x - 3 = 0$ then what could it be? For example, how can you tell that neither $+2$ nor -2 could be a root?

You have now met the main techniques to find factors, but you should always remember that some polynomials will not have any **real** roots, whilst others will have real roots, but they will not be integers or even rational numbers.

Exercise 6C

1. Show that
   (a) $(x-1)$ is a factor of $x^4+3x^3-2x^2+5x-7$
   (b) $(x+1)$ is a factor of
   $x^5-7x^4-8x^3+2x^2+x-1$
   (c) $(x-3)$ is a factor of x^4-2x^3-7x-6
   (d) $(x-2)$ is a factor of $2x^6+5x^4-27x^3+8$
   (e) $(x+10)$ is a factor of
   $5x^5+23x^4-269x^3-90x+100$
   (f) $(x-8)$ is a factor of
   $x^{10}-9x^9+8x^8-x^2+x+56$

2. Three of the polynomials below have a linear factor from the list on the right. Match the factors to the polynomials. The 'odd one out' does have a simple linear factor, but not one in the list. Find the factor.

A: x^4-5x^2+3x-2	1: $x+1$
B: $x^5-4x^4+2x^3-8x^2-x+4$	2: $x-2$
C: x^3-4x^2-6x-1	3: $x+3$
D: $x^3+12x^2-43x+6$	

3. Find a linear factor for each of these polynomials.
   (a) x^3+4x^2+x-6
   (b) x^4-6x-4
   (c) $x^4-6x^3-8x^2+6x+7$
   (d) $2x^4+x^3-6x^2-x+1$
   (e) $3x^3+10x^2-11x+2$

6.4 Solving higher order equations

The next activity shows how you can tackle higher order equations using the factor theorem.

Activity 6 Solving a quartic equation

The equation $x^4-4x^3-13x^2+4x+12=0$ is an example of a **quartic** equation, a polynomial equation of degree 4.

Find a simple linear factor of the quartic, and use this factor to complete a statement of this type :

$$x^4-4x^3-13x^2+4x+12=(x-\alpha)\times(\text{cubic})$$

Hence find all the solutions of the quartic equation and sketch the graph of $y=x^4-4x^3-13x^2+4x+12$ without the help of a calculator or computer.

Activity 6 shows how the factor theorem can help solve higher order equations. However, it may have taken you some time! Probably the longest part was finding the cubic factor. When the same process is applied to, say,

$$x^6 - x^4 + 17x - 14 = (x+2)$$ (polynomial of degree 5) it will take longer still.

Fortunately there are ways of finding the polynomial factor which are more efficient. Two methods will be introduced here. You may, later on, like to reflect on how they are essentially the same method expressed differently.

The first method involves juggling with coefficients, and is best demonstrated by example.

Consider the quartic $x^4 + 5x^3 + 7x^2 + 6x + 8$. $x = -2$ is a zero of this function and hence $(x+2)$ is a factor. The cubic factor can be found thus :

$$\begin{aligned}
&x^4 + 5x^3 + 7x^2 + 6x + 8 \\
&\quad = x^3(x+2) + 3x^3 + 7x^2 + 6x + 8 \\
&\quad = x^3(x+2) + 3x^2(x+2) + x^2 + 6x + 8 \\
&\quad = (x^3 + 3x^2)(x+2) + x(x+2) + 4x + 8 \\
&\quad = (x^3 + 3x^2 + x)(x+2) + 4(x+2) \\
&\quad = (x^3 + 3x^2 + x + 4)(x+2)
\end{aligned}$$

Hence the cubic factor is $x^3 + 3x^2 + x + 4$.

Example

Find the missing expression here :

$$x^4 - 3x^3 + 5x^2 + x - 4 = (x-1)(\qquad)$$

Solution

$$\begin{aligned}
&x^4 - 3x^3 + 5x^2 + x - 4 \\
&\quad = x^3(x-1) - 2x^3 + 5x^2 + x - 4 \\
&\quad = x^3(x-1) - 2x^2(x-1) + 3x^2 + x - 4 \\
&\quad = (x^3 - 2x^2)(x-1) + 3x(x-1) + 4x - 4 \\
&\quad = (x^3 - 2x^2 + 3x)(x-1) + 4(x-1) \\
&\quad = (x^3 - 2x^2 + 3x + 4)(x-1)
\end{aligned}$$

so the missing expression is $x^3 - 2x^2 + 3x + 4$.

The second method is sometimes called 'long division of polynomials'. The statement

$$x^4 + 5x^3 + 7x^2 + 6x + 8 = (x+2)(\text{cubic})$$

can be written instead in this form

$$\frac{x^4 + 5x^3 + 7x^2 + 6x + 8}{x+2} = \text{cubic polynomial}$$

The cubic polynomial is thus the result of dividing the quartic by $(x+2)$. This division can be accomplished in a manner very similar to long division of numbers.

$$\begin{array}{r|l}
 & x^3 + 3x^2 + x + 4 \\
\hline
x+2 & x^4 + 5x^3 + 7x^2 + 6x + 8 \\
 & \underline{x^4 + 2x^3} \\
 & \quad 3x^3 + 7x^2 \\
 & \quad \underline{3x^3 + 6x^2} \\
 & \qquad x^2 + 6x \\
 & \qquad \underline{x^2 + 2x} \\
 & \qquad\quad 4x + 8 \\
 & \qquad\quad \underline{4x + 8} \\
 & \qquad\qquad 0
\end{array}$$

So $x^4 + 5x^5 + 7x^2 + 6x + 8 = (x+2)\left(x^3 + 3x^2 + x + 4\right)$,

the same answer is obtained as on the previous page by the 'juggling' method.

Activity 7 Long division

When $x = 10$, the quotient $\dfrac{x^4 + 5x^3 + 7x^2 + 6x + 8}{x+2}$ becomes the division sum $15\ 768 \div 12$. Evaluate this by long division (**not** short division). Discuss the resemblance between your sum and the algebraic long division above. Try putting other values of x into the quotient (e.g. $x = 9$, $x = -5$).

Example

Evaluate the quotient $\dfrac{x^4-5x^3-2x^2+25x-3}{x-3}$.

Solution

$$
\begin{array}{r|l}
 & x^3-2x^2-8x+1 \\
\hline
x-3 & x^4-5x^3-2x^2+25x-3 \\
 & \underline{x^4-3x^3} \\
 & \quad -2x^3-2x^2 \\
 & \quad \underline{-2x^3+6x^2} \\
 & \qquad -8x^2+25x \\
 & \qquad \underline{-8x^2+24x} \\
 & \qquad\qquad x-3 \\
 & \qquad\qquad \underline{x-3} \\
 & \qquad\qquad\quad 0
\end{array}
$$

The quotient is therefore x^3-2x^2-8x+1.

Example

Work out the missing cubic factor in this statement:

$$4x^4-11x^2+15x-18=(2x-3)(\ldots\ldots)$$

Solution

Note that there are no terms in x^3. To simplify the division, a term $0x^3$ is included.

$$
\begin{array}{r|l}
 & 2x^3+3x^2-x+6 \\
\hline
2x-3 & 4x^4+0x^3-11x^2+15x-18 \\
 & \underline{4x^4-6x^3} \\
 & \quad 6x^3-11x^2 \\
 & \quad \underline{6x^3-9x^2} \\
 & \qquad -2x^2+15x \\
 & \qquad \underline{-2x^2+3x} \\
 & \qquad\qquad 12x-18 \\
 & \qquad\qquad \underline{12x-18} \\
 & \qquad\qquad\quad 0
\end{array}
$$

The missing cubic factor is therefore $2x^3+3x^2-x+6$.

Exercise 6D

1. Find these quotients by long division
   (a) $7982 \div 26$
   (b) $22149 \div 69$
   (c) $45694 \div 134$
   (d) $55438 \div 106$
2. Use the 'juggling' method to find the missing factors
   (a) $x^4 + 8x^3 + 17x^2 + 12x + 18 = (x+3)(......)$
   (b) $x^4 - 5x^3 - x^2 + 25 = (x-5)(......)$
   (c) $x^3 + 4x^2 - 8 = (x+2)(......)$
3. Use long division to find the missing factors
   (a) $x^4 + 6x^3 + 9x^2 + 5x + 1 = (x+1)(......)$
   (b) $x^4 + 7x^3 - 39x - 18 = (x+6)(......)$
   (c) $2x^3 - 4x^2 - 7x + 14 = (x-2)(......)$
   (d) $9x^5 + 9x^4 - 16x^3 + 11x + 2 = (3x+2)(......)$
4. Evaluate these quotients
   (a) $\dfrac{x^3 + 6x^2 - 6x + 7}{x^2 - x + 1}$ (b) $\dfrac{2x^4 + 5x^3 - 5x - 2}{x^2 + 3x + 2}$
   (c) $\dfrac{x^4 - 6x^3 + 4x^2 - 6x + 3}{x^2 + 1}$ (d) $\dfrac{x^4 + 2x^3 - 2x - 4}{x^3 - 2}$

6.5 Factorising polynomials

In the next examples, you will see how to factorise polynomials of degrees higher than two.

Example

Factorise the quartic $x^4 - 4x^3 - 7x^2 + 34x - 24$ as fully as possible and hence solve the equation $x^4 - 4x^3 - 7x^2 + 34x - 24 = 0$.

Solution

$x = 1$ is a zero of the quartic, so $(x-1)$ is a factor.

Long division yields

$$x^4 - 4x^3 - 7x^2 + 34x - 24 = (x-1)(x^3 - 3x^2 - 10x + 24)$$

Now factorise the cubic; since $x = 2$ is a zero of the cubic, $(x-2)$ is a factor.

Long division gives

$$x^3 - 3x^2 - 10x + 24 = (x-2)(x^2 - x - 12)$$

The quadratic $x^2 - x - 12$ factorises easily to give $(x-4)(x+3)$

The full factorisation of the original quartic is therefore

$$(x-1)(x-2)(x-4)(x+3)$$

and the solutions to the equation are thus

$$x = -3, 1, 2 \text{ and } 4.$$

Example

Solve the equation $x^4+3x^3-11x^2-19x-6=0$.

Solution

First, factorise the quartic as far as possible.

$x=-1$ is a zero so $(x+1)$ is a factor.

By long division

$$x^4+3x^3-11x^2-19x-6=(x+1)(x^3+2x^2-13x-6)$$

$x=3$ is a zero of the cubic so $(x-3)$ is a factor.

$$x^3+2x^2-13x-6=(x-3)(x^2+5x+2)$$

The quadratic x^2+5x+2 has no straightforward linear factors, but the equation $x^2+5x+2=0$ does have solutions, namely -4.56 and -0.438.

The quartic thus factorises to $(x+1)(x-3)(x^2+5x+2)$ yielding solutions $x=-4.56, -1, -0.438$ and 3.

Exercise 6E

1. Solve the following quartic equations :

   (a) $x^4+8x^3+14x^2-8x-15=0$

   (b) $x^4+8x^3-13x^2-32x+36=0$

2. Solve the quintic equation

   $x^5-8x^3+6x^2+7x-6=0$

6.6 Remainders

In the previous section you saw how to divide a polynomial by a factor. These ideas will be extended now to cover division of a polynomial by an expression of the form $(x-\alpha)$.

If $(x-\alpha)$ is not a factor, there will be a remainder.

Activity 8

(a) Carry out the long division $\dfrac{x^3+5x^2-2x+1}{x+1}$

(b) If you substitute $x=9$ in this quotient it becomes $1117 \div 10$. Carry out the division. Repeat with $x=10, 11, 12$. Do not use a calculator. Comment on your answers and how they correspond to part (a).

As with numbers, long divisions of polynomials often leave remainders. For example $(x+3)$ is **not** a factor of x^3+6x^2+7x-4, and so long division will yield a remainder.

$$
\begin{array}{r}
x^2+3x-2 \\
x+3\,\overline{\big)\,x^3+6x^2+7x-4} \\
\underline{x^3+3x^2}\qquad\qquad \\
3x^2+7x\qquad \\
\underline{3x^2+9x}\qquad \\
-2x-4 \\
\underline{-2x-6} \\
2
\end{array}
$$

One way of expressing this might be to write

$$\frac{x^3+6x^2+7x-4}{x+3} = x^2+3x-2, \text{ rem } 2$$

but the normal method is to write either

$$x^3+6x^2+7x-4 = (x^2+3x-2)(x+3)+2$$

or $$\frac{x^3+6x^2+7x-4}{x+3} = x^2+3x-2+\frac{2}{x+3}$$

Activity 9 Remainders

(a) Substitute any positive integer for x in the quotient

$\frac{x^3+6x^2+7x-4}{x+3}$. Verify by division that the remainder is 2.

Try some negative values of x (except -3) and comment.

(b) Find the remainder when x^3+6x^2+x-7 is divided by $x+1$.

(c) Find the remainder for the division $(x^3+x^2-4x+8)\div(x-2)$.

Exercise 6F

1. Find the remainders

   (a) when $x^2-15x+10$ is divided by $(x-5)$

   (b) when $x^3+4x^2-7x+10$ is divided by $(x+3)$

2. Use Question 1 to complete these statements

   (a) $\frac{x^3+4x^2-7x+10}{x+3}=(......)+\frac{......}{x+3}$

   (b) $x^2-15x+10=(......)(x-5)+.....$

6.7 Extending the factor theorem

Activity 10

(a) How can you tell that $(x-1)$ is not a factor of $x^3-7x+10$?

(b) When the division is carried out a statement of the form

$$x^3-7x+10=(x-1)Q(x)+R$$

will result, where $Q(x)$ is a quadratic function and R the remainder. Without doing the division, calculate R. (Hint : choose a suitable value of x to substitute in the equation above.)

(c) Without doing the division, calculate the remainder when $x^4+3x^3-5x+10$ is divided by $x+2$.

Activity 10 illustrates the result known as the **remainder theorem**.

> If $P(x)$ is a polynomial of degree n then
>
> $$P(x) = (x-\alpha)Q(x) + R$$
>
> where $Q(x)$ is a polynomial of degree $n-1$ and $R = P(\alpha)$.

The useful fact that $R = P(\alpha)$ can be demonstrated simply by considering what happens to the equation $P(x) = (x-\alpha)Q(x) + R$ when $x = \alpha$ is substituted into it.

Does this provide a proof of the remainder theorem?

*Activity 11 Division by quadratics

Carry out the algebraic division

$$\frac{x^4 + 3x^3 - 10x^2 - 26x + 28}{x^2 + 2x - 3}$$

Can you suggest a remainder theorem for division by quadratics? Can you generalise to division by any polynomial?

Exercise 6G

1. Work out the remainder when

   (a) $x^2 + 5x - 7$ is divided by $(x-2)$;

   (b) $x^4 - 3x^2 + 7$ is divided by $(x+3)$;

   (c) $5x^3 + 6x^2 + 2x - 3$ is divided by $(x+5)$.

2. When the quadratic $x^2 + px + 1$ is divided by $(x-1)$ the remainder is 5. Find p.

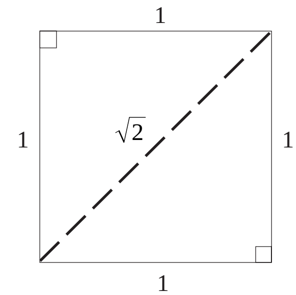
3. $x^2 + px + q$ divides exactly by $(x-5)$ and leaves remainder –6 when divided by $(x+1)$.

   Find p and q.

4. Find the linear expressions which yield a remainder of 6 when divided into $x^2 + 10x + 22$.

6.8 Rationals and irrationals

Until about the 5th century AD it was firmly believed that whole numbers and their ratios could be used to describe any quantity imaginable. In other words, that the set Q of rational numbers contained every number possible. Gradually, though, mathematicians became aware of 'incommensurable quantities', quantities that could not be expressed as the ratio of two integers. Such numbers are called **irrational numbers**.



The origin of this concept is uncertain, but one of the simplest examples of an irrational number arises from Pythagoras' Theorem. This theorem gives the length of the diagonal of the unit square as $\sqrt{2}$.

The proof that $\sqrt{2}$ is irrational is one of the most famous proofs in all mathematics. It employs a technique called '*reductio ad absurdum*' (reduction to the absurd) the proof begins by assuming that $\sqrt{2}$ is rational and then shows that this assumption leads to something impossible (absurd). See if you can follow the reasoning:

Suppose $\sqrt{2}$ is rational.

This means that $\sqrt{2} = \frac{p}{q}$, where p and q are integers with no common factor, and $q \neq 0$.

$$\sqrt{2} = \frac{p}{q} \;\Rightarrow\; \frac{p^2}{q^2} = 2 \;\Rightarrow\; p^2 = 2q^2$$

$\Rightarrow$ p^2 is even $\Rightarrow p$ is even.

Hence p can be written as $2r$, where r is an integer.

Since p is even, q must be odd as they have no common factors.

$$p^2 = 2q^2 \;\Rightarrow\; 4r^2 = 2q^2 \;\Rightarrow\; q^2 = 2r^2$$

$\Rightarrow$ q^2 is even $\Rightarrow q$ is even.

q is thus seen to be both odd **and** even, which is impossible. The original assumption that $\sqrt{2}$ is rational must therefore be false.

Hence $\sqrt{2}$ is irrational.

Other examples of irrationals are π and any square root like $\sqrt{5}$. Still more examples can be constructed from these, e.g. $\sqrt{2}+1, 6\pi$, etc. The set of rational **and** irrational numbers is called the set of real numbers, denoted by $\mathbb{R}$.

Activity 12 True or false?

Decide whether these statements are true or false

(a) (i) rational + rational = rational

(ii) rational + irrational = irrational

(iii) irrational + irrational = irrational

(b) (i) rational × rational = rational

(ii) rational × irrational = rational

(iii) irrational × irrational = rational

(c) 'Between any two rational numbers there is another rational number'.

(d) 'Between any two irrational numbers, there is another irrational number'.

A different way of looking at rational and irrational numbers comes from considering equations.

To solve any linear equation involving integer coefficients, the set of rationals is sufficient.

e.g. $71x + 1021 = 317 \Rightarrow x = -\dfrac{704}{71}$

However, Q is **not** sufficient to solve every polynomial of degree 2 and higher. While some do have rational solutions,

e.g. $x^2 - 7x + 12 = 0 \Rightarrow x = 3 \text{ or } 4$

in general, they do not.

e.g. $x^2 - 3x + 1 = 0 \Rightarrow x = \frac{1}{2}(3 \pm \sqrt{5})$.

In fact, there is still a further class of numbers; these are the **transcendental** numbers. If you are interested, find out what these are by consulting a mathematical dictionary.

Irrational expressions like $\sqrt{5}$ are called **surds**. Surds cannot be expressed as ratios of natural numbers.

Activity 13 Handling surds

(a) Use a calculator to verify that $\sqrt{8} = 2\sqrt{2}$. Explain why this is true.

(b) What surd can be written as $3\sqrt{2}$?

(c) Express $\sqrt{12}$ as a multiple of $\sqrt{3}$.

Use a calculator to check your answer.

A method of manipulating surds is used to 'rationalise' the denominators in expressions like

$$\frac{1}{\sqrt{5}+1}$$

To rationalise a denominator means literally to turn an irrational denominator into a rational one.

Activity 14 How to rationalise a denominator

(a) What is $(\sqrt{5}+1)(\sqrt{5}-1)$?

(b) Multiply the fraction $\left(\frac{1}{\sqrt{5}+1}\right)$ by $\left(\frac{\sqrt{5}-1}{\sqrt{5}-1}\right)$.

(c) Explain why $\frac{1}{\sqrt{5}+1} = \frac{\sqrt{5}-1}{4}$.

(d) What is $(\sqrt{10}-2)(\sqrt{10}+2)$?

(e) Write $\frac{3}{(\sqrt{10}-2)}$ as a fraction with a rational denominator.

Example

(a) $\sqrt{20} = \sqrt{4\times 5} = \sqrt{4}\times\sqrt{5} = 2\sqrt{5}$

(b) $\sqrt{216} = \sqrt{4\times 54} = 2\sqrt{54}$

But $\sqrt{54}$ can itself be simplified :

$$2\sqrt{54} = 2\sqrt{9\times 6} = 2\sqrt{9}\times\sqrt{6} = 6\sqrt{6}$$

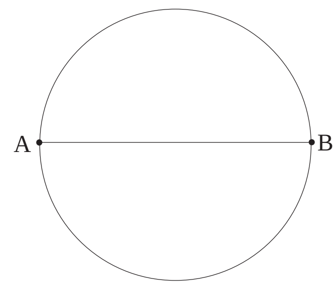
Example

(a) $\dfrac{5}{\sqrt{6}-1}=\left(\dfrac{5}{\sqrt{6}-1}\right)\left(\dfrac{\sqrt{6}+1}{\sqrt{6}+1}\right)=\dfrac{5(\sqrt{6}+1)}{5}=\sqrt{6}+1$

(b) $\dfrac{1}{\sqrt{11}+\sqrt{7}}=\left(\dfrac{1}{\sqrt{11}+\sqrt{7}}\right)\left(\dfrac{\sqrt{11}-\sqrt{7}}{\sqrt{11}-\sqrt{7}}\right)=\dfrac{\sqrt{11}-\sqrt{7}}{4}$

Activity 15 Fence posts

The diagram shows a circular field cut in half by the diameter AB. The owner of the field wants to build two fences, one round the circumference of the circle, the other across the diameter. Fence posts are placed at A and B and further posts spaced equally along the diameter.



Explain why the owner must use a different measure of equal spacing when fence posts are put around the circumference. ('Spacing' is to be taken to mean 'distance apart round the circumference').

Activity 16 $\sqrt{3}$

Prove that $\sqrt{3}$ is irrational, using the same method of proof as was used for $\sqrt{2}$.

Can the same method be used to 'prove' that $\sqrt{4}$ is irrational?

Exercise 6H

1. Write down an irrational number
   (a) between 3 and 4;
   (b) between 26 and 27;
   (c) between -6 and -5.
2. Simplify the surds
   (a) $\sqrt{52}$ (b) $\sqrt{75}$ (c) $\sqrt{120}$ (d) $\sqrt{245}$
3. Use a similar technique to simplify
   (a) $\sqrt[3]{16}$ (b) $\sqrt[3]{54}$ (c) $\sqrt[4]{48}$
4. Rationalise the denominators in these expressions
   (a) $\dfrac{1}{\sqrt{2}-1}$ (b) $\dfrac{3}{\sqrt{21}-3}$ (c) $\dfrac{2}{\sqrt{5}-\sqrt{2}}$
   (d) $\dfrac{3}{\sqrt{2}}$ (e) $\dfrac{5}{\sqrt{14}-2}$

6.9 Miscellaneous Exercises

1. Find a linear factor for each of these polynomials :

   (a) $x^4-3x^3-10x^2-x+5$

   (b) $x^5+3x^4+x^3+5x^2+12x-4$

2. Solve these equations :

   (a) $x^4-3x^3-10x^2-x+5=0$

   (b) $x^4+2x^3-67x^2-128x+192=0$

3. Copy and complete these identities :

   (a) $x^3-7x^2-x-6=(x-7)(\quad)+(\quad)$

   (b) $\dfrac{x^5+5x^4-3x^2+2x+1}{x+2}=(\quad)+\dfrac{(\quad)}{x+2}$

4. What is the remainder when

   (a) $x^4-5x^2+12x-15$ is divided by $x+3$;

   (b) $x^3-5x^2-21x+7$ is divided by $x-10$?

5. $x^3+ax^2+5x-10$ leaves remainder 4 when divided by $x+2$. Find a.

6. Find the linear expressions which leave remainder 14 when divided into $x^2-5x-10$.

7. A quadratic function is exactly divisible by $x-2$, leaves remainder 12 when divided by $x+1$ and remainder 8 when divided by $x-3$. What is the quadratic function?

8. Find the linear expressions which leave remainder –8 when divided into $x^3-12x^2+17x+22$.

9. Simplify these, where possible

   (a) $\dfrac{x^2-1}{2(x+1)}$ (b) $\dfrac{6a^3}{2a+a^2}$

   (c) $\dfrac{m^2-4}{2m}$ (d) $\dfrac{3x^2+5x-2}{3x+6}$

   (e) $\dfrac{x^3-1}{x-1}$

10. Find the remainder when

    $p(x)=2x^3-x^2-2x+1$

    is divided by

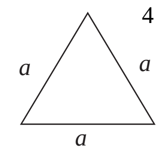
    (a) $x-2$ (b) $x+1$

    Factorise $p(x)$ completely. (AEB)

11. The polynomial $p(x)\equiv 2x^3-9x^2+kx-13$, when k is a constant, has $(2x-1)$ as a factor. Determine the value of k. (AEB)

12. An equilateral triangle has sides of length a.

    Show that the area is $\dfrac{a^2\sqrt{3}}{4}$.



13. (a) Show that

    (i) $(p+4q)$ is a factor of $p^2+2pq-8q^2$ and find the other linear factor;

    (ii) $(x-2y)$ is a factor of $x^3+x^2y-7xy^2+2y^3$ and find the quadratic factor;

    (iii) $(a+3b)$ is a factor of $2a^3+7a^2b-9b^3$ and find the two other linear factors.

    (b) Factorise these expressions completely

    (i) $x^2-2ax-3a^2$

    (ii) $p^2+10pq-24q^2$

    (iii) $a^3-2a^2b-11ab^2+12b^3$

14. Factorise the expression

    $16x^5-81x$

    as completely as possible.