5 SOLVING PROBLEMS

Objectives

After studying this chapter you should

- have gained experience in formulating several types of problem in mathematical terms;
- appreciate how algebra can be used to solve problems;
- have improved your fluency in handling a variety of algebraic expressions;
- be able to solve several types of equations and inequalities.

5.0 Introduction

A more traditional title for this chapter might have been 'Elementary Algebraic Techniques', however, this would not tell the whole story. It is certainly true that you will have the opportunity to practice and develop your algebraic skills here, but perhaps the more important aim is to emphasise algebra as an effective problem-solving aid.

Algebra cannot solve every problem. Even where it can, there are sometimes alternatives. Nevertheless, algebra is an efficient way of communicating ideas and formulating problems; once a problem has been expressed in algebraic form, finding the answer to the problem is often much simpler.

5.1 Setting up and solving linear problems

Do not be put off by the simplicity of this problem.

> A coal merchant charges £8 per tonne of coal plus a fixed delivery charge of £5. How much coal can be delivered for £50?
>
> What is the answer if the merchant delivers only in 100 kg bags?

The purpose of beginning with such an easy problem is to demonstrate the general method of solving problems algebraically. Algebraic problem solving normally falls into three stages:

Stage 1 Translate the problem into a mathematical problem.

Stage 2 Solve the mathematical problem.

Stage 3 Translate the answer back into the terms of the original problem.

The simple problem above could be solved algebraically like this:

$$\left.\begin{array}{l}\text{Let the number of tonnes delivered be } x.\\ \text{Total cost of } x \text{ tonnes} = £(5+8x).\\ \text{But the total cost} = £50,\end{array}\right\}\text{Stage 1}$$

so $5+8x=50$.

$$\left.\begin{array}{ll}5+8x=50 & \\ \Rightarrow & 8x=45\\ \Rightarrow & x=5.625\end{array}\right\}\text{Stage 2}$$

$$\left.\begin{array}{l}x=5.625,\\ \text{so the amount of coal that can be delivered is}\\ 5.625 \text{ tonnes or } 5 \text{ tonnes } 625\text{kg}.\end{array}\right\}\text{Stage 3}$$

(Check the answer : $5+(8\times 5.625)=5+45=50.$)

The last part of the problem could be put slightly differently. If the merchant delivered only in 100 kg bags then an answer of 5.625 tonnes is inadmissible, and clearly the required answer, by common sense, is 5.6 tonnes.

For the mathematics to come up with the proper answer the problem needs to be set up as an inequation, or inequality, rather than as equation.

Let the number of bags delivered be x.

Total cost of x bags $= £(5+0.8x)$

Total cost must not exceed £50, and x must be a natural number (i.e. a positive whole number):

$$5 + 0.8x \le 50,\ \ x \in \mathbb{N} \quad \text{(the set of natural numbers)}$$
$$\Rightarrow \quad 0.8x \le 45,\ \ x \in \mathbb{N}$$
$$\Rightarrow \quad x \le 56.25,\ \ x \in \mathbb{N}$$

The last line suggests directly that the largest possible value of x must be 56.

Since x represents the number of 100 kg bags the answer to the problem must now be **5.6 tonnes**.

Note also the importance of choosing the right domain for x. The inequality combined with the condition '$x \in \mathbb{N}$' yields the correct solution.

Below are three further examples. Before reading the solutions you should try to solve the problems by yourself, or in a group; then compare your solutions to the versions given in the text.

Make sure you set out each solution clearly. Try to follow the format of the worked examples given above.

Example

A coach travels along the M3 from Winchester to London. It sets off at 10.30 am and travels at a constant 65 mph. A car makes the same journey, travelling 10 mph faster but leaving 5 minutes later. When does the car overtake the coach?

Solution

Let t represent the number of minutes since the coach left Winchester, and let T be the number of minutes since the car left Winchester.

$$\text{Distance travelled by coach} = \frac{65t}{60} \text{ miles.}$$

$$\text{Distance travelled by car} = \frac{75T}{60} \text{ miles.}$$

The car left 5 minutes later than the coach, so when $T = 0$, $t = 5$. Hence $T = t - 5$.

The car overtakes the coach when they have travelled the same distance.

$$\frac{65}{60}t = \frac{75}{60}T$$

$$\Rightarrow \quad 65t = 75(t-5)$$

$$\Rightarrow \quad 13t = 15(t-5)$$

$$\Rightarrow \quad 13t = 15t - 75$$

$$\Rightarrow \quad 2t = 75$$

$$\Rightarrow \quad t = 37.5$$

So the car overtakes the coach $37\frac{1}{2}$ minutes after the coach's departure, i.e. at $11.07\frac{1}{2}$ am.

Example

Tickets for a pantomime cost £8 for adults and £5 for children. A party of 25 pays £143 in total. How many adults were there in the party?

Solution

Let the number of adults be x. Then the number of children must be $25 - x$.

Cost of x adult tickets and $(25 - x)$ children tickets

$$= £[8x + 5(25 - x)]$$

But the total cost was £143, so

$$8x + 5(25 - x) = 143$$

$$\Rightarrow \quad 3x + 125 = 143$$

$$\Rightarrow \quad 3x = 18$$

$$\Rightarrow \quad x = 6$$

Hence there were **6 adults** in the party.

(Check: 6 adult tickets and 19 child tickets cost $6 \times 8 + 19 \times 5 = 143$.)

Example

Alan has a non-interest current account at his bank from which mortgage repayments of £250 are made monthly. At the beginning of the year the account contains £3000. Barbara has a similar account: her monthly payments are £370 and her account contains £4150 at the start of the year. After how many payments does Alan's account contain more money than Barbara's?

Solution

Let the number of payments be *n*. After *n* payments:

Alan's account contains £$(3000-250n)$

Barbara's account contains £$(4150-370n)$

Alan's account has more money than Barbara's.

$$
\begin{aligned}
\Rightarrow \quad 3000-250n &> 4150-370n,\ n \in \mathbb{N} \\
\Rightarrow \quad -250n &> 1150-370n,\ n \in \mathbb{N} \\
\Rightarrow \quad 120n &> 1150,\ n \in \mathbb{N} \\
\Rightarrow \quad n &> 9.583,\ n \in \mathbb{N}
\end{aligned}
$$

The last line suggests that the lowest possible value of *n* is 10. Hence Alan's account first has more money than Barbara's after **10 payments**.

Activity 1 Using simultaneous equations

The solution to the second example might have begun like this: Let the party consist of *x* adults and *y* children.

$$\text{Total cost} = 8x + 5y = 143$$

$$\text{Total number of people} = x + y = 25$$

Solve these simultaneous equations and check that the answer is the same as before.

(If you need to, revise how to solve simultaneous equations.)

Similarly the first example might have been solved as the answer to these simultaneous equations.

$$\left.\begin{aligned} 13t &= 15T \\ t-5 &= T \end{aligned}\right\}$$

Check that the answers obtained satisfy these questions.

All the equations in this section are called **linear** as they contain terms in *x* and *y* but not in x^2, y^2, xy, etc.

Exercise 5A below contains some more examples for you to try. The exercise begins with a brief opportunity to revise the solution of linear equations and inequalities.

Exercise 5A

1. Solve these equations. If the answers are not exact, give them to 3 s.f.

   (a) $1400-3a=638$ (b) $\dfrac{4(b+45)}{5}=24$

   (c) $16c+24=21c-481$ (d) $6(35-d)=d+19$

   (e) $\dfrac{13.2e-10.7}{4}=6.1$ (f) $\frac{1}{2}(3f+11)=6f-47$

2. Solve the following pairs of simultaneous equations

   (a) $\begin{cases}5p+3q=612\\ p+q=150\end{cases}$ (b) $\begin{cases}13m-5n=434\\ m-n=50\end{cases}$

   (c) $\begin{cases}9k+6l=306\\ 5k-2l=154\end{cases}$ (d) $\begin{cases}y=2x-4\\ y=5x+11\end{cases}$

   (e) $\begin{cases}p=2q-30\\ q=6p+16\end{cases}$ (f) $\begin{cases}u+3v+10=0\\ 3u=2v+1\end{cases}$

3. Solve these inequalities

   (a) $15a+365\le 560$

   (b) $61000-103b<62648$

   (c) $\frac{2}{3}(11c-76)\ge 8,\quad c\in\mathbb{N}$

   (d) $5(d+6.3)>2d+3.2$

   (e) $-\left(\dfrac{187+65e}{3}\right)\le 12e-500$

   (f) $16.3(f-5)\ge 4(7.9-3.2f),\ f\in\mathbb{N}$

4. A school is looking to furnish two new classrooms. Tables cost £11 each and chairs £6. Each table will have two chairs except the two teachers' tables which will have one chair each. The total budget is £770.

   Assuming the budget is spent exactly, how many chairs and tables will the school buy?

5. An amateur operatic society needs to buy 20 vocal scores for their latest show. Hardback copies cost £14.50, paperback copies £9.20. £250 has been put aside to buy these scores. What is the maximum number of hardback copies they can buy and satisfy their needs?

6. At 6.00 pm a house's hot water tank contains only 10 litres of water, while the cold tank contains 100 litres. The cold water tank is emptying at a rate of 12 litres/min, half of this amount flowing into the hot tank. Water from the mains supply flows into the cold water tank at 3 litres/min. After how many minutes will the two tanks contain equal amounts of water?

7. Andrea is trying to save money on electricity bills. At present she estimates her family uses 8000 units of electricity per year at 6.59 pence per unit. They also have to pay an annual standing charge of £36.28.

   She is told that installing an 'Economy 7' system might save money. This means that off-peak units are charged at 2.63 pence per unit and the rest at 6.59p, but there is an additional annual standing charge of £10.04. She wants to know how many of the 8000 units would have to be off-peak units in order to save at least £50 per year. Solve her problem.

5.2 Revision

Section 5.1 concentrated more on the process of formulating problems in algebraic terms than on the algebra itself. Hence the algebra was uncomplicated. However, this is often not true, so this section is intended as an opportunity to revise some algebraic techniques you may have studied before. The worked examples should serve as a reminder of them before you tackle Exercise 5B.

Example

Multiply out (i.e. write without brackets):

(a) $p(p-7q)$ (b) $2mn(m^2-3n^2)$

(c) $(x+3)(x-5)$ (d) $(k^2-2l)(k+l)$.

Solution

(a) $p(p-7q) = p^2-7pq$

(b) $2mn(m^2-3n^2) = 2m^3n-6mn^3$

(c) $(x+3)(x-5) = x(x-5)+3(x-5)$

$$= x^2-5x+3x-15$$

$$= x^2-2x-15$$

(d) $(k^2-2l)(k+l) = k^2(k+l)-2l(k+l)$

$$= k^3+k^2l-2kl-2l^2.$$

Example

Factorise (i.e. simplify using brackets) :

(a) $2u^2+6u$ (b) ab^2-3a^3b.

Solution

(a) $2u^2+6u = 2u(u+3)$

(b) $ab^2-3a^3b = ab(b-3a^2)$.

Example

Simplify the expression

$$(p+3)(p-1)-(p+2)(p-3).$$

Solution

$$\begin{aligned}
&(p+3)(p-1)-(p+2)(p-3) \\
&\qquad = (p^2+2p-3)-(p^2-p-6) \\
&\qquad = p^2+2p-3-p^2+p+6 \\
&\qquad = 3p+3 \\
&\qquad = 3(p+1).
\end{aligned}$$

Example

Factorise these :

(a) x^2+5x+6 (b) $x^2-9x+20$

(c) $x^2-3x-40$ (d) $x^2+2x-99$.

Solution

All these expressions are quadratic and will factorise to the form $(x+m)(x+n)$. Now

$$(x+m)(x+n) = x^2 + (m+n)x + mn,$$

so to find the appropriate values of m and n compare $x^2 + (m+n)x + mn$ with the quadratic required.

(a) $x^2 + 5x + 6$:

$m+n = 5$ and $mn = 6$, so m and n must be 2 and 3, and

$$x^2 + 5x + 6 = (x+2)(x+3)$$

(b) $x^2 - 9x + 20$:

Two numbers are required whose **product** is $+20$ and whose **sum** is -9. From the list on the right the required numbers are clearly -4 and -5, and

$$x^2 - 9x + 20 = (x-4)(x-5)$$

$$\begin{aligned} 20 &= 1\times 20 = (-1)\times(-20) \\ &= 2\times 10 = (-2)\times(-10) \\ &= 4\times 5 = (-4)\times(-5) \end{aligned}$$

(c) $x^2 - 3x - 40$

Similarly from the list of factors of 40 the factorisation must be $(x-8)(x+5)$

(d) $x^2 + 2x - 99 = (x+11)(x-9)$.

Example

Factorise

(a) $x^2 - 9$ (b) $4a^2 - 9b^2$

(c) $2x^2 - 9x + 4$ (d) $3x^2 - 2x - 16$

(e) $3x^2 + 6x - 24$.

Solution

(a) and (b) are the difference of two squares. You may recall that

$$p^2 - q^2 = (p+q)(p-q)$$

and so, by comparison,

(a) $x^2 - 9 = (x+3)(x-3)$

(b) $4a^2 - 9b^2 = (2a)^2 - (3b)^2$

$$= (2a+3b)(2a-3b)$$

(c) and (d) are more difficult. Some methods are given below, but the technique is still best learned through experience.

$4 = 1\times 4 = (-1)\times(-4)$
$= 2\times 2 = (-2)\times(-2)$

(c) $2x^2 - 9x + 4$ must factorise to the form $(2x+m)(x+n)$. Now $(2x+m)(x+n) = 2x^2 + (m+2n)x + mn$. So the product of m and n must be 4, but notice that the n is doubled in the $(m+2n)x$ term. Looking at the list opposite the factorisation will be $(2x-1)(x-4)$.

(d) $3x^2 - 2x - 16$.

Similar techniques as before, but note that in multiplying out $(3x+m)(x+n)$ the 'n' is multiplied by '$3x$' to help give a term $(m+3n)x$.

$3x^2 - 2x - 16 = (3x-8)(x+2)$

(e) looks similar to (d), but notice that all the terms have a factor of 3, so

$$\begin{aligned} 3x^2 + 6x - 24 &= 3(x^2 + 2x - 8) \\ &= 3(x+4)(x-2). \end{aligned}$$

Exercise 5B

1. Multiply out :

   (a) $2(5x+7y)$ (b) $a(a-b)$

   (c) $3lm(6l-5m)$ (d) $p^2(2p-3q^2+1)$

   (e) $h^3k^3(2hk+3hk^2)$.

2. Multiply out :

   (a) $(x+1)(x+2)$ (b) $(x+3)(x-5)$

   (c) $(x-8)(x-2)$ (d) $(x+6)(x-5)$

   (e) $(x-10)(x+7)$.

3. Multiply out :

   (a) $(2x-7)(x-3)$ (b) $(2x+3)(x+1)$

   (c) $(3x+20)(x-8)$ (d) $(a-2b)(2a+b)$

   (e) $(2m^2-n)(m+5n^2)$.

4. Multiply out :

   (a) $(x+1)^2$ (this is not x^2+1)

   (b) $(p-3)^2$ (c) $(x-10)^2$

   (d) $(x+a)^2$ (a is any number)

   (e) $(2x-5)^2$.

5. Simplify these :

   (a) $2a(a-b)+b(2a-b)$

   (b) $(y-2)(y+3)+(2y+5)(y+1)$

   (c) $(n+4)^2-(n-1)(n+7)$

   (d) $5(m-6)(m+2)-2(2m+3)(m-7)$

   (e) $(u+2v)(u-v)-(4u-v)(u+3v)$.

6. Factorise these expressions :

   (a) $6x+15$ (b) u^2-3u

   (c) $3p^2+24p$ (d) $12a^2b^3-6ab^2$

   (e) $5x^2y+15y-35$.

7. Factorise these :

   (a) x^2+6x+8 (b) x^2+x-30

   (c) $x^2-7x+10$ (d) x^2+5x+4

   (e) $x^2-3x-70$ (f) $x^2-10x+9$

   (g) $x^2+6x-16$ (h) $x^2-5x-84$.

8. Factorise these :

   (a) x^2-16 (b) x^2-25x

   (c) $2x^2+7x+3$ (d) $2x^2-7x-4$

   (e) $3x^2-11x-20$ (f) $2x^2-4x-6$

   (g) $3x^2-4x-20$ (h) $100x^2-64$.

9. Factorise these :

   (a) $5x^2+3x-2$ (b) $4x^2+5x-6$

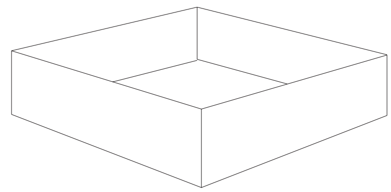
   (c) $4x^2-4x-3$ (d) $6x^2+15x-36$

   (e) $6x^2+5x-25$ (f) $12x^2-7x-10$.

5.3 Setting up and solving quadratic equations

Activity 2 A problem of surface area

The diagram shows an open-topped box with a square base. The sides of the box are 3 cm high.



The box is to be made from a total of 160 cm^2 of card. What size must the square base be to maximise the volume contained? Try to set out an algebraic solution as in the previous section.

Solving the above problem algebraically leads to a quadratic equation, that is, an equation involving a quadratic function. You may have already covered how to solve this type of equation. The following worked example and activities should help to refresh your memory.

Example

Solve the equation $x(x+6)=16$.

Solution

$$\begin{aligned} & x(x+6)=16 \\ \Rightarrow\quad & x^2+6x=16 \\ \Rightarrow\quad & x^2+6x-16=0. \end{aligned}$$

$x^2+6x-16$ can be factorised to give $(x+8)(x-2)$. Hence the equation becomes

$$\begin{aligned} & (x+8)(x-2)=0 \\ \Rightarrow\quad & x=-8 \text{ or } 2. \end{aligned}$$

Example

Solve the equation $3x^2+10x=25$.

Solution

The procedure is exactly the same, though the factorisation is more difficult.

$$
\begin{aligned}
& 3x^2 + 10x = 25 \\
\Rightarrow \quad & 3x^2 + 10x - 25 = 0 \\
\Rightarrow \quad & (3x-5)(x+5) = 0 \\
\Rightarrow \quad & x = \tfrac{5}{3} \text{ or } -5.
\end{aligned}
$$

Activity 3 Solutions and graphs

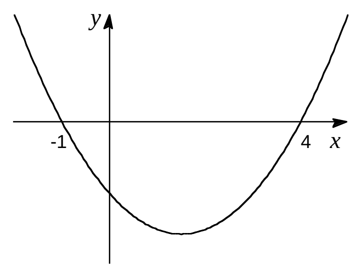
(a) Sketch the graph of $y = (x+9)(x-3)$
At what points does the curve cross the horizontal axis?

(b) At what point will the graph of $y = (x+m)(x+n)$ cross the horizontal axis?

(c) Sketch the graph of $y = (2x-1)(x+2)$. Explain why the curve crosses the $x-\text{axis}$ where it does.

(d) Sketch the graphs of these functions all on one diagram, without using any electronic aids.

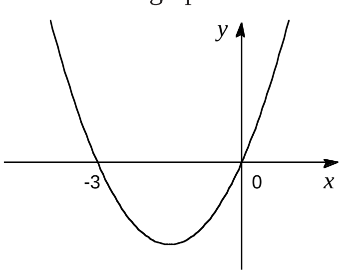
$$
\begin{aligned}
y &= (x-6)(x+2) \\
y &= 2(x-6)(x+2) \\
y &= \tfrac{1}{3}(x-6)(x+2) \\
y &= -(x-6)(x+2)
\end{aligned}
$$



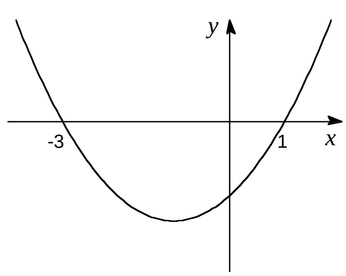
(e) Give a possible equation for the quadratic graph on the right. How many possibilities are there? What else would you need to know to narrow these possibilities down to one?

Activity 4 Match the graphs to the functions

Shown are eight quadratic functions, numbered 1 to 8, and five graphs, lettered A to E. Each graph corresponds to one of the functions. Decide which function goes with which graph. Draw sketches of the graphs of the functions that are not used.



A.



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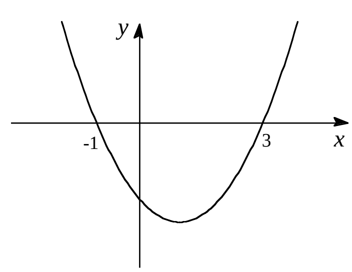
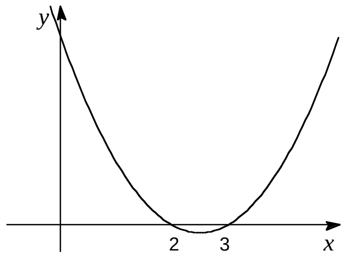
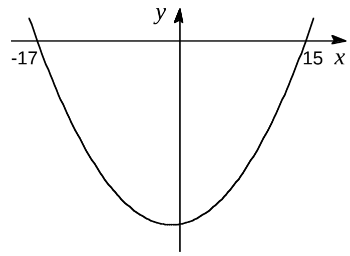


C.

1. $x^2 + 3x$
2. $(x-1)(x+3)$
3. $(x+17)(x-15)$
4. $(x+2)(x+3)$
5. $(x-17)(x+15)$
6. $x^2 - 5x + 6$
7. $(x+1)(x-3)$
8. $x(x-3)$.



D.

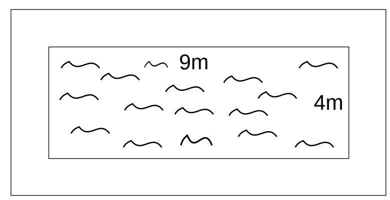


E.

Quadratic equations often arise when solving problems connected with area. (Can you think why?) When a quadratic equation does present itself, it is important to bear in mind the domain of the variable concerned: in general, quadratic equations have two solutions, but it is not necessarily true that both of them are solutions to the original problem, as you will see in the next example.

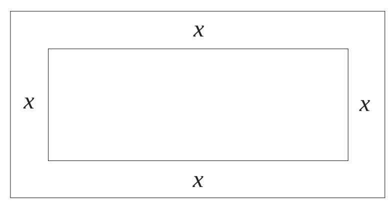
Example

Joe wishes to make a gravel path around his rectangular pond. The path must be the same width all the way round, as shown in the diagram. The pond measures 4 m by 9 m and he has enough gravel to cover an area of 48 m^2. How wide should the path be?



Solution

Let the width of the footpath be x metres. The diagram shows that the area of the path is



$$\begin{aligned}&(9+2x)(4+2x)-36\\&\quad=36+26x+4x^2-36\\&\quad=4x^2+26x.\end{aligned}$$

Since the required area is 48 m^2, then $4x^2 + 26x = 48$.

To this equation must be added the condition that $x > 0$.

$$\begin{aligned}&& 4x^2+26x=48, &\qquad x>0\\ \Rightarrow && 2x^2+13x-24=0, &\qquad x>0\\ \Rightarrow && (2x-3)(x+8)=0, &\qquad x>0\\ \Rightarrow && x=\tfrac{3}{2}\ \text{or}\ -8, &\qquad x>0\\ \Rightarrow && x=\tfrac{3}{2}. &\end{aligned}$$

Hence the width of the path must be 1.5m, since the other value, $x = -8\text{m}$, is unrealistic, and does not satisfy the condition $x > 0$.

Exercise 5C

1. Solve these quadratic equations by factorising

   (a) $x(x+3)=4$ (b) $x(x+5)=50$

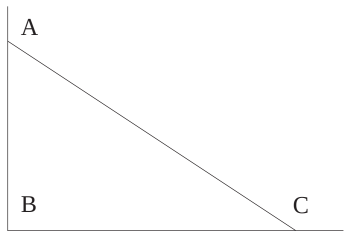
   (c) $x^2=x+72$ (d) $x^2+77=18x$

   (e) $x^2+50x+96=0$ (f) $x^2-50x+600=0$

   (g) $2x^2-13x+15=0$ (h) $x(2x-1)=21$

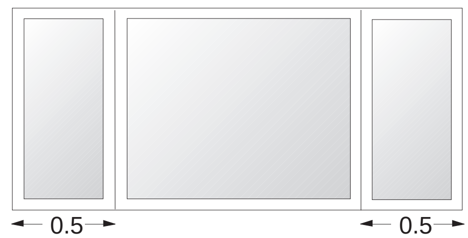
   (i) $3x^2-6x=45$ (j) $3x^2-14=x$.

2. A triangular flowerbed is to be dug in the corner of a rectangular garden. The area is to be 65 m^2.



   The owner wants the length BC of the bed to be 3 m longer than the width AB. Find what the length and width should be.

3. A window is designed to be twice as wide as it is high. It is to be made up of three parts: a central, fixed part and two identical rectangles on either side of width 0.5 metres.



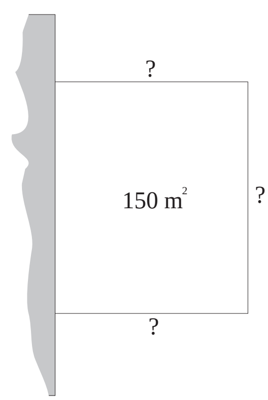
   Find the height of the window, if the central part is to have an area of 3 m^2.

5.4 Approximate solutions

In practical problems which lead to solving quadratic equations, it is not always possible to factorise the equation.

Activity 5 A fencing problem

50 m of fencing is being used to enclose a rectangular area, with a long straight wall as one of the sides of the rectangle. What dimensions give an area of 150 m^2?



As Activity 5 will have illustrated, not all quadratic equations can be solved by factorisation. Another example is the equation

$$x^2-15x+40=0$$

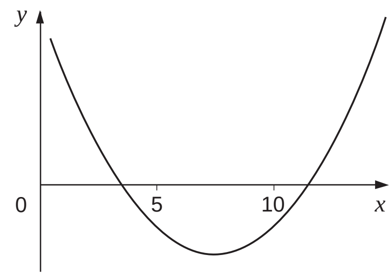
which cannot be factorised.

You may know how to deal with such equations already - an algebraic method will be discussed in Section 5.5 - but you can use a combination of graphical or trial and improvement methods to find approximate solutions.

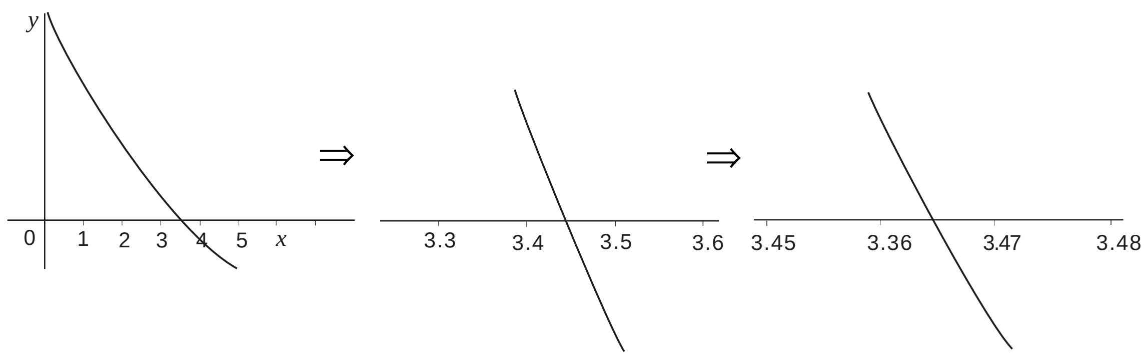
The **graphical method** can be undertaken with the help of a graphic calculator or with pencil and graph paper. Either way, draw the graph of

$$y = x^2 - 15x + 40$$

and find where it crosses the horizontal axis.



If you are doing this by hand you will need to be accurate; if you are using a computer or calculator, greater accuracy can be obtained by zooming in on each solution in turn.



You may be familiar with the **trial and improvement** technique. It is best explained by example. The results of successive trials can be set out in a table.

x	$x^2 - 15x + 40$
0	40
1	26
2	14
3	4
4	–4

This shows that there must be a solution between $x = 3$ and $x = 4$, probably near 3.5. Now look more closely between $x = 3$ and $x = 4$.

3.4	0.56	root between $x = 3.4$ and 3.5
3.5	–0.25	
3.46	0.0716	root between $x = 3.46$ and 3.47 being closer to 3.47
3.47	–0.0091	

The solution is 3.47 to 3 s.f. Greater accuracy can be achieved by continuing the process as long as necessary. This, of course, is only one solution. The process must be repeated to find the other. (Try it yourself.)

Activity 6

The quadratic $x^2+10x-20$ cannot be factorised using whole numbers but can be factorised approximately using decimals. Find its factors correct to 2 d.p.'s.

Does a quadratic equation always have precisely two solutions?

Activity 7 Computer program

Devise a computer program to solve a quadratic equation by trial and improvement.

Exercise 5D

1. Use a graphical approach to solve these equations to 3 s.f.

   (a) $x(2x-3)=9$ (b) $x^2+x=14$

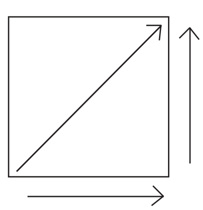
   (c) $2x(20-x)=195$ (d) $300-x^2=6x$.

2. Solve these equations to 3 s.f. using trial and improvement.

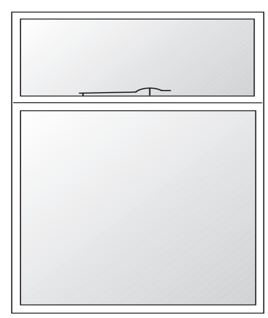
   (a) $x(x+3)=8$ (b) $7x-x^2=5$

   (c) $x^2+8x+11=0$ (d) $x(x+25)=8000$.

3. The distance across the diagonal of a square field is 50 m shorter than going round the perimeter. To the nearest metre, find the length of the side of the field.



4. An architect decides that the smallest room in the house needs a window with area 4000 cm^2.



The window will be in two parts: a square part at the bottom, and a rectangular part above it hinged at the top and with height 25 cm. Find the width the window must be to satisfy these requirements. Answer to the degree of accuracy you think is appropriate to the problem.

5.5 Investigating quadratic functions

Graphical and numerical methods are of great importance in solving equations. Their big advantages are that they can be used to solve practically any equation, and solve them to any degree of accuracy. However it can be a slow process, and where quadratic equations cannot be solved by factorising there is another method of solution that is often faster.

Consider the equation $x^2 - 4x - 3 = 0$.

Activity 8 Completing the square

(a) Another way of expressing the function $x^2 - 4x - 3$ is to write it in the form

$$(x - p)^2 - q.$$

Multiply out this expression and equate it to $x^2 - 4x - 3$ to find the values of p and q.

(b) Having put the function in this new form, now solve the equation $x^2 - 4x - 3 = 0$.

Remember that you should obtain two solutions.

(c) Solve the equations

(i) $x^2 - 2x - 1 = 0$, and

(ii) $x^2 - 3x - 5 = 0$

by a similar method.

The method used in the activity above is called **completing the square**. Any quadratic can be written in this form. For example,

$$\begin{aligned} 2x^2 - 12x + 15 &= 2\left(x^2 - 6x + \frac{15}{2}\right) \\ &= 2\left((x-3)^2 - 9 + \frac{15}{2}\right) \\ &= 2\left((x-3)^2 - \frac{3}{2}\right). \end{aligned}$$

$(x-3)^2$: the 3 comes from half 'x'coefficient

-9 : introduced to cancel out 3^2

This quadratic $2x^2 - 12x + 15 = 0$ can now be solved since

$$2\left((x-3)^2 - \frac{3}{2}\right) = 0$$

$$\Rightarrow \quad (x-3)^2 - \frac{3}{2} = 0$$

$$(x-3)^2 = \frac{3}{2}$$

$$x-3 = \pm\sqrt{\frac{3}{2}}$$

$$x = 3 \pm\sqrt{\frac{3}{2}}.$$

Example

By completing the square, solve

(a) $x^2+6x+4=0$ (b) $x^2-3x+1=0$

(c) $5x^2-6x-9=0$.

Solution

(a) $x^2+6x+4 = (x+3)^2-9+4$

$= (x+3)^2-5.$

Hence the equation $x^2+6x+4=0$ can be rewritten

$$(x+3)^2-5=0$$

$$\Rightarrow \quad (x+3)^2=5$$

$$\Rightarrow \quad x+3=\pm\sqrt{5}$$

$$\Rightarrow \quad x = -3\pm\sqrt{5}$$

$= -5.24$ or -0.764 to 3 s.f.

(b) $x^2-3x+1=(x-1.5)^2-(1.5)^2+1$

$=(x-1.5)^2-1.25.$

Hence $x^2-3x+1=0$ can be re-written

$$(x-1.5)^2-1.25=0$$

$$\Rightarrow \quad (x-1.5)^2=1.25$$

$$\Rightarrow \quad x-1.5=\pm\sqrt{1.25}$$

$$\Rightarrow \quad x = 1.5\pm\sqrt{1.25}$$

$= 0.382$ or 2.62 to 3 s.f.

(c) $5x^2-6x-9=5[x^2-1.2x-1.8]$

$=5[(x-0.6)^2-(0.6)^2-1.8]$

$=5[(x-0.6)^2-2.16].$

Hence $5x^2 - 6x - 9 = 0$ can be re-written

$$(x - 0.6)^2 - 2.16 = 0$$

$$\Rightarrow \quad x = 0.6 \pm \sqrt{2.16}$$

$$= -0.870 \text{ or } 2.07 \text{ to 3 s.f.}$$

The technique of completing the square has wider applications than just solving equations. It is, for example, useful in helping to sketch curves of quadratic functions.

Look, for example, at the function $5x^2 - 6x - 9$ (see (c) above). Completing the square, as suggested by Activity 8, can be seen as the splitting up of a quadratic function into several simpler ones. In this case, $f(x) = 5x^2 - 6x - 9$ is equivalent to the transformation

$$a : x \mapsto x - 0.6$$

followed by $b : x \mapsto x^2$

followed by $c : x \mapsto x - 2.16$

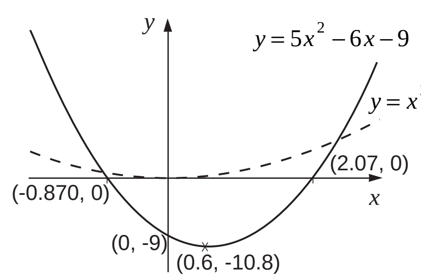
followed by $d : x \mapsto 5x$.

These four functions gradually build up the expression required:

$$\begin{aligned} a(x) &= x - 0.6 \\ ba(x) &= (x - 0.6)^2 \\ cba(x) &= (x - 0.6)^2 - 2.16 \\ dcba(x) &= 5[(x - 0.6)^2 - 2.16] \\ &= 5x^2 - 6x - 9. \end{aligned}$$

Activity 9

In earlier chapters you found out how simple graphs were affected by simple transformations. Explain why the graph of $y = 5x^2 - 6x - 9$ looks like the sketch opposite.



Activity 10 Transformations of $y = x^2$

(a) Using the graph of $y = x^2$, sketch the graphs of these functions, to the same scale.

(i) $y = (x - 3)^2 + 2$ (ii) $y = (x + 5)^2 - 8$

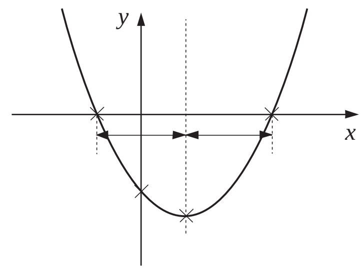
(iii) $y = 3[(x - 1)^2 + 1]$ (iv) $y = x^2 - 8x + 10$

(v) $y = 2x^2 + 5x - 3$.

Do not use any graph-plotting package except to check your answers. Label the crucial points in the same way as the sketch above.

(b) Explain how the minimum point of the quadratic curve $y = x^2 - 10x + 20$ can be found without having to draw the curve itself.

Activity 11



On the right is a sketch of the graph of

$$y = (x - p)^2 - q.$$

Copy it and mark on

(a) the co-ordinates of the points marked with crosses;

(b) the line of symmetry of the graph, with its equation;

(c) the distances marked with arrows, $\longleftrightarrow$.

Activity 12 Maximum values

(a) Plot the graphs of the functions

$$y = 4 + 2x - x^2$$

$$y = 10 - 5x - 2x^2$$

and give an explanation for their general shape.

(b) Show that

$$4 + 2x - x^2 = 5 - (x-1)^2$$

and rewrite $10 - 5x - 2x^2$ in a similar way.

(c) What are the maximum values of these two functions? How can you tell without drawing a graph?

As you can see another application of completing the square is that of finding the maximum or minimum values of quadratic functions.

Example

Find the minimum value of the function $x^2 + 3x + 7$.

Solution

$$x^2 + 3x + 7 = (x + 1.5)^2 + 4.75$$

So the minimum value must be 4.75, which occurs when $x = -1.5$.

Example

Find the value of x that maximises the function $100 + 50x - 2x^2$ and find this maximum value.

Solution

$$\begin{aligned} 100 + 50x - 2x^2 &= -2\left[x^2 - 25x - 50\right] \\ &= -2\left[(x - 12.5)^2 - 206.25\right] \\ &= 412.5 - 2(x - 12.5)^2. \end{aligned}$$

Thus the maximum value is 412.5 achieved when $x = 12.5$.

Exercise 5E

1. Solve the quadratic equations to 3 s.f.

   (a) $a^2 - 4a - 7 = 0$ (b) $b^2 - 12b + 30 = 0$

   (c) $c^2 - c - 1 = 0$ (d) $2d^2 + 20d = 615$

   (e) $2e^2 - 8e + 1 = 0$ (f) $5f^2 + 4f = 13$

   (g) $53 + 30g - g^2 = 0$ (h) $1225 + 46h - 2h^2 = 0$.

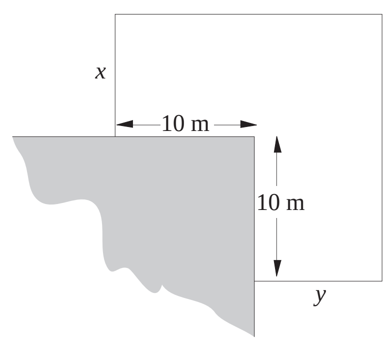
2. (a) What value of x maximises the function
   $x^2 + 40x + 25$?

   (b) What is the minimum value taken by the function
   $x^2 + 9x + 14$?

   (c) What is the maximum value of $46 - 24x - x^2$?

   (d) What is the maximum value of $25 - 60x - 2x^2$ and what value of x makes this function a maximum?

3. 100 m of fencing is used to make an enclosure as shown in the diagram, jutting out from the corner of a rectangular building.



   If x and y are as marked, show that $y = 40 - x$ and hence

   (a) find x if the area of the enclosure is to be 600 m^2;

   (b) find the maximum area that can be enclosed.

5.6 Quadratic solution formula

Completing the square is a technique that can be used to solve any quadratic equation. In many cases it is easy to use. However, in practice quadratic equations can sometimes have awkward coefficients and 'completing the square' leads to some unhappy arithmetic; for example,

$$7.8x^2 - 11.2x - 4.9 = 0$$

$$\Rightarrow \quad 7.8\,[x^2 - 1.4359x - 0.6282] = 0 \quad \text{(working to 4 d.p.'s)}$$

$$\Rightarrow \quad (x - 0.7179)^2 - 1.1437 = 0$$

$$\Rightarrow \quad x = 0.7179 \pm \sqrt{1.1437}$$

$$= -0.351 \text{ or } 1.79 \text{ to 3 s.f.}$$

For cases like this there is a formula which will enable you to calculate the answer more quickly.

Activity 13 Finding a formula

(a) If $x^2 + bx + c = 0$, show that $\left(x + \frac{b}{2}\right)^2 + c - \left(\frac{b}{2}\right)^2 = 0$ and hence that

$$x = -\frac{b}{2} \pm \sqrt{\frac{b^2}{4} - c}\,.$$

Can you see why this formula is the same as

$$x = \frac{-b \pm \sqrt{b^2 - 4c}}{2} \;?$$

(b) Now extend this process to solve the general quadratic equation

$$ax^2 + bx + c = 0\,.$$

The two solutions to $ax^2 + bx + c = 0$ can be shown to be given by

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

Example

Solve the equation $7.8x^2 - 11.2x - 4.9 = 0$.

Solution

Use the formula with $\begin{cases} a = 7.8 \\ b = -11.2 \\ c = -4.9 \end{cases}$

Solutions are $x = \dfrac{11.2 \pm \sqrt{(-11.2)^2 - 4 \times (7.8) \times (-4.9)}}{2 \times 7.8}$

$= \dfrac{11.2 \pm \sqrt{278.32}}{15.6}$

$= -0.351$ or 1.79 to 3 s.f.

Activity 14

(a) Complete the square for the function $x^2 - 6x + 10$. Sketch its graph, labelling the minimum point clearly.

(b) Try to solve the equation $x^2 - 6x + 10 = 0$ using the formula.
What happens? Explain your answer graphically.

(c) Try to solve the equation $x^2 - 12x + 36 = 0$ using the formula.
Use a graph to explain what happens.

(d) What general statements can you make about quadratic equations?

The quantity $(b^2 - 4ac)$ is clearly of importance in the behaviour of quadratics. It is referred to as the **discriminant** of the function.

What happens if $b^2 - 4ac = 0$ or $b^2 - 4ac < 0$?

Exercise 5F

1. Use the formula $\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$ to solve these equations to 3 s.f.:

   (a) $0.7a^2-0.2a-0.8=0$

   (b) $17b^2+12b-1=0$

   (c) $2.05c^2=4.79c+12.26$

   (d) $1500d^2-17000d=100000$

   (e) $\frac{1}{2}e^2-\frac{1}{4}e+\frac{1}{45}=0$

   (f) $2\pi f^2=150f-400$.

2. Find the discriminant of each of the following equations. How many solutions does each have? (You are not expected to find the solutions.)

   (a) $x^2+x+10=0$

   (b) $x^2+10x-1=0$

   (c) $3x^2-x-15=0$

   (d) $4x^2-36x+81=0$

   (e) $1.72x^2+5.71x+4.68=0$

   (f) $9x^2+60x+100=0$.

3. A stone is hurled vertically upwards at a speed of 21 metres per second. Its height above the ground in metres is given by the formula

   $$2+21t-5t^2 .$$

   (a) After how many seconds does the stone hit the ground?

   (b) What is the maximum height reached by the stone?

4. The surface area of a circular cylinder of radius r and height h is $2\pi r(r+h)$. What radius, to 3 s.f., will give a cylinder of height 15 cm a surface area of 600 cm^2?

5.7 Equations with fractions

You may at this moment be using A4 paper. Paper sizes in the 'A' series have a special property.

If you fold a piece of A4 paper in half you get two pieces of size A5 (see diagram).

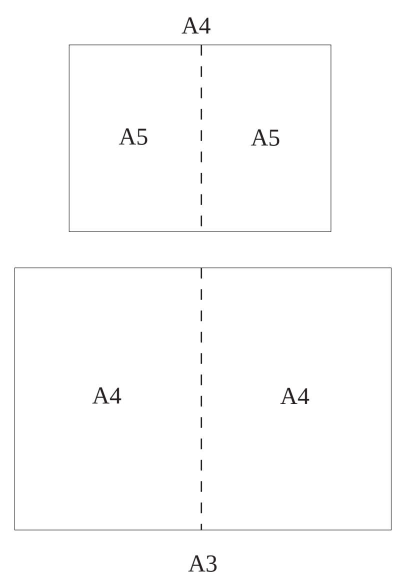
If you put two A4 sheet together side by side, then the resulting size is A3.

Two A3 sheets side by side make an A2 sheet and so on.

All the rectangles in the 'A' series are similar, that is, the ratio

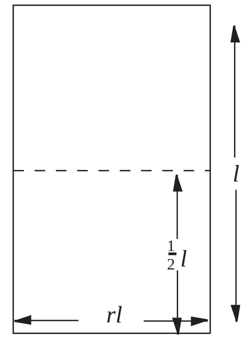
$$\frac{\text{shortest side}}{\text{longest side}}$$

is the same for each size. But what is this ratio?



The problem can be solved algebraically. Let r denote the required ratio, and let l stand for the length of a sheet of A4 paper. The width is therefore rl.

Cutting the sheet in half makes an A5 sheet, with the same ratio. For the A5 sheet:



$$\text{shortest side} = \tfrac{1}{2}l \qquad \text{longest side} = rl.$$

Hence $$\frac{\frac{1}{2}l}{rl} = r, \quad 0 < r < 1.$$

This is an equation involving an algebraic fraction. The best way to solve it is to multiply both sides by rl, the denominator of the fraction, so that the fraction disappears. This leaves

$$\tfrac{1}{2}l = r^2 l, \quad 0 < r < 1$$

$$\Rightarrow \quad \tfrac{1}{2} = r^2, \quad 0 < r < 1$$

$$\Rightarrow \quad r = +\sqrt{\tfrac{1}{2}} = 0.707 \text{ to } 3\,\text{s.f.}$$

The answer must now be validated.

Measure a piece of A4 paper to check this answer.

Algebraic fractions often turn up when solving problems connected with ratio or rates. Here is another example.

Example

Two tests for a typing certificate are being designed. Each test consists of a passage to be typed as quickly and accurately as possible. The tests will follow on, one after the other, without a break. The following criteria should also be used:

- between them the tests should last 25 minutes;
- the anticipated typing speed for test 1 is 36 words per minute and for test 2, 45 words per minute;
- both tests should contain the same number of words.

How many words should each test contain?

Solution

Let W stand for the number of words in each test. The total time is 25 minutes, so

$$\frac{W}{45} + \frac{W}{36} = 25,$$

which must be solved to find W.

In any equation involving fractions an efficient way to start is to multiply through by the denominators so that no fractions are left. In doing this, take care to multiply each separate term by the same number. So to solve the equation above you first multiply the equation by 45, to give

$$45\times\frac{W}{45}+45\times\frac{W}{36}=45\times 25$$

$$\Rightarrow \qquad W+\frac{45}{36}W=1125$$

$$\Rightarrow \qquad W+\frac{5}{4}W=1125.$$

Now multiply both sides by 4:

$$4W+5W=4500$$
$$\Rightarrow \qquad 9W=4500$$
$$\Rightarrow \qquad W=500.$$

Example

Solve the equation $\frac{x}{2}=\frac{10}{x+1}$.

Solution

$$\frac{x}{2}=\frac{10}{x+1}$$
$$\Rightarrow \qquad x=\frac{20}{x+1} \qquad \text{(multiply both sides by 2)}$$
$$\Rightarrow \qquad x(x+1)=20 \qquad \text{(multiply both sides by } (x+1)\text{)}$$
$$\Rightarrow \qquad x^2+x-20=0$$
$$\Rightarrow \qquad (x+5)(x-4)=0$$
$$\Rightarrow \qquad x=-5 \text{ or } 4.$$

Example

If $\frac{24}{x}+\frac{30}{x-1}=10$, find x.

Solution

$$\frac{24}{x}+\frac{30}{x-1}=10$$

$\Rightarrow \quad 24+\dfrac{30x}{x-1}=10x$ (multiply both sides by x)

$\Rightarrow \quad 24(x-1)+30x=10x(x-1)$ (multiply both sides by $(x-1)$)

$\Rightarrow \quad 24x-24+30x=10x^2-10x$

$\Rightarrow \quad 10x^2-64x+24=0$

$\Rightarrow \quad 5x^2-32x+12=0$

$\Rightarrow \quad (5x-2)(x-6)=0$

$\Rightarrow \quad x=\frac{2}{5}$ or 6.

Activity 15 Spot the deliberate mistakes

These two solutions contain deliberate errors; spot them, correct them, and find the right answers.

(a) $\dfrac{24}{x+1}=\dfrac{x}{6}$

$\Rightarrow \quad \dfrac{24}{1}=\dfrac{x^2}{6}$

$\Rightarrow \quad 6\times 24=x^2$

$\Rightarrow \quad 144=x^2$

$\Rightarrow \quad x=\pm 12.$

(b) $\dfrac{8}{x}+\dfrac{10}{x-3}=3$

$\Rightarrow \quad 8+\dfrac{10x}{x-3}=3$

$\Rightarrow \quad 8(x-3)+10x=3$

$\Rightarrow \quad 18x-24=3$

$\Rightarrow \quad x=1.5.$

Exercise 5G

1. Solve the equations

(a) $\frac{x}{7}+\frac{2x}{3}=17$ (b) $\frac{x}{9}-\frac{x}{12}=1$

(c) $\frac{x}{8}=\frac{18}{x}$ (d) $\frac{3}{x+1}=\frac{1}{2}$

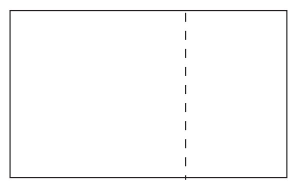
(e) $\frac{7}{x+1}=\frac{8}{x-2}$ (f) $\frac{2x-3}{5}=\frac{27}{x}$

(g) $\frac{x+1}{3}=\frac{18}{x-2}$ (h) $\frac{6}{x+2}+\frac{4}{x-2}=1$

(i) $\frac{18}{2x-1}-\frac{15}{x}+1=0$ (j) $\frac{x+1}{4}-\frac{20}{x-5}=0$.

2. Foolscap paper obeys the following property.

   If the sheet is folded to make a square and a rectangle, the rectangular part is similar to the foolscap sheet itself, that is, for both rectangles the ratio $\frac{\text{longest side}}{\text{shortest side}}$ is the same. Find this ratio.



3. 'Petrol is 75p per gallon more expensive now than it was in 1981. You can buy as much petrol now for £132 as you could for £87 in 1981'. Assuming these statements are true, find the cost of petrol in 1981.

4. Sandra buys some 'fun sized' chocolate bars as prizes for her son's birthday party. She spends £16.80 on them in total.

   She recalls that she spent exactly the same amount of money on them for last year's party, but this year the same amount bought 20 bars fewer, since the price had gone up by 2p a bar.

   What is the current cost of a 'fun-sized' chocolate bar?

5.8 Inequalities

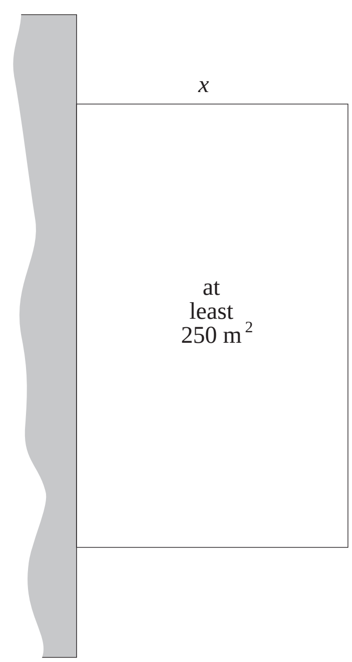
Here is a variation on a problem posed in Activity 5 of Section 5.4

Activity 16 A quadratic inequality

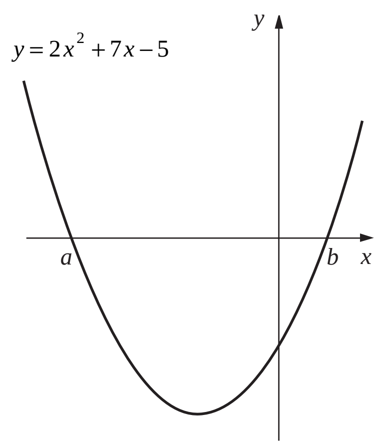
A farmer uses 50 m of fencing to form a rectangular enclosure with one side against a wall as shown opposite. The area of th enclosure must be at least 250 m^2.

Show that $x^2-25x+125\leq 0$, where x metres is the width of t enclosure.

Find the range of values of x that satisfy this inequality. (You might find it helpful to draw a graph of the function)

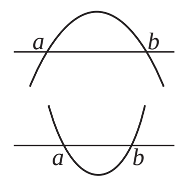


Quadratic inequalities can be solved by a mixture of algebraic and graphical means. For example, to solve $2x^2 + 7x - 5 > 0$, you might first consider the graph of the function $y = 2x^2 + 7x - 5$, shown on the right. The solution to the inequality is clearly the two regions $x < a$ and $x > b$, where a and b are the two solutions to the quadratic equation $2x^2 + 7x - 5 = 0$.



The values of a and b can be found either by completing the square or by using the formula. This gives $a = 4.11$ and $b = 0.608$. Hence the solution to the equality is $x < -4.11$ or $x > 0.608$.

This solution could have been found without drawing a graph at all. Once you know where the quadratic function is zero there are two possibilities.



Either - the function is positive between these two values, negative elsewhere;

Or - the function is negative between these two values, positive elsewhere.

You can quickly discover which of these is true by choosing a number between a and b and seeing whether the function is positive or negative. In the example above, look to see what happens at $x = 0$:

$$x = 0 \Rightarrow y = 2x^2 + 7x - 5 = -5$$

so function is negative between $x = -4.11$ and $x = 0.608$ and the required solution is either side of these values.

Example

Solve the inequality $4(x^2 - 250) \le 65x$

Solution

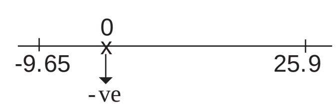
$$4(x^2 - 250) \le 65x$$

$$\Rightarrow \quad 4x^2 - 65x - 1000 \le 0.$$

The function $4x^2 - 65x - 1000$ is zero at $x = -9.65$ and 25.9 to 3 s.f.

Choose a value between -9.65 and 25.9, say $x = 0$.

When $x = 0$, $4x^2 - 65x - 1000 = -1000$ so the function is negative between -9.65 and 25.9.



The required solution is between these two values:

$$-9.65 \le x \le 25.9.$$

Two alternative methods are illustrated in the next two activities.

Activity 17

(a) Consider the inequality $x^2 > 9$. Why is the answer $x > 3$ incomplete? What is the full solution?

(b) Now consider the inequality

$$x^2 - 6x - 11 \geq 0.$$

Complete the square and use this form of the function to solve the inequality directly.

Activity 18

In the inequality $2x^2 - 7x - 39 < 0$ the quadratic function can be factorised. Rewrite the inequality in the form

$$(\qquad)\;(\qquad) < 0.$$

If the product of two brackets is negative, as required by this inequality, what can you say about the two factors?

Hence solve the inequality.

Two further worked examples are given below, one for each of these alternative methods. After that there follows an exercise for you to practice these techniques.

Example

Solve $x^2 + 8x + 9 < 0$.

Solution

$x^2 + 8x + 9$ cannot be factorised, but it is easy to complete the square.

$$\begin{aligned}
& x^2 + 8x + 9 < 0 \\
\Rightarrow\quad & (x+4)^2 - 7 < 0 \\
\Rightarrow\quad & (x+4)^2 < 7 \\
\Rightarrow\quad & -\sqrt{7} < x + 4 < +\sqrt{7} \\
\Rightarrow\quad & -\sqrt{7} - 4 < x < +\sqrt{7} - 4.
\end{aligned}$$

Example

Solve $x^2 - 3x - 28 \geq 0$.

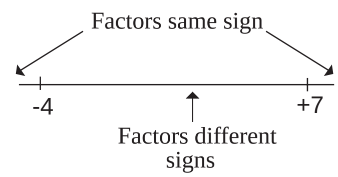
Solution

$x^2 - 3x - 28$ can be factorised

$$x^2 - 3x - 28 \geq 0$$
$$\Rightarrow \quad (x-7)(x+4) \geq 0.$$

The two factors can be either both positive or both negative, as the diagram suggests, so the solution must be

$$x \leq -4,\ x \geq 7.$$



Exercise 5H

1. Solve these by factorising

   (a) $x^2 - 5x - 66 \geq 0$

   (b) $3x^2 + 1 < 4x$

   (c) $(x-7)(x+5) \geq 28$

   (d) $x^2 + 110x + 3000 > 0$.

2. Solve these inequalities by completing the square.

   (a) $x^2 + 2x > 4$

   (b) $x^2 - 8x + 10 \leq 0$

   (c) $x^2 + 24x + 100 < 0$

   (d) $x(x-100) \geq 2000$.

3. Solve these inequalities by whatever method you choose.

   (a) $x^2 - 7x \leq 60$

   (b) $x^2 + 12x + 28 > 0$

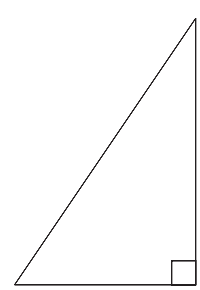
   (c) $x(50-3x) < 200$

   (d) $6x^2 + 11x \leq 350$

   (e) $\dfrac{14x - 3x^2}{9} \geq 1$

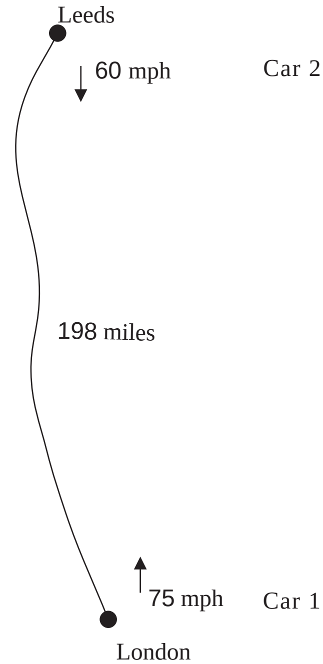
   (f) $0.7x^2 - 3.9x + 2.5 < 0$.

4. In a right-angled triangle the hypotenuse is more than double the shortest side. The third side of the triangle is 2 cm longer than the shortest side. What values can the shortest side take? (Remember the domain when setting up the problem).



5.9 Modulus sign

The use of the modulus sign will be introduced through this problem.



> Two cars are driving in opposite directions on the motorway. One starts from London at a constant speed of 75 mph; the other starts at the same time from Leeds, 198 miles away, at a constant 60 mph.
>
> Both cars are equipped with two-way radios but need to be within 10 miles of each other to make contact. When can they contact one another?

If t is the number of minutes after the journeys start then

distance of car 1 from London

$$= \frac{75t}{60} \text{ miles}$$
$$= \frac{5t}{4} \text{ miles.}$$

Distance of car 2 from Leeds $= \dfrac{60t}{60} = t$ miles, so distance of car 2 from London $= (198 - t)$ miles.

Their distance apart is found by subtracting one distance from the other, i.e. distance apart in miles is

$$= 198 - t - \frac{5t}{4}$$
$$= 198 - \frac{9t}{4}.$$

So, for example, after one hour $(t = 60)$ the cars are 63 miles apart. But when $t = 120$, the formula gives -72. This means that the cars are 72 miles apart, but that car 2 is now closer to London than car 1, i.e. the cars have crossed.

There is an agreed convention for getting round this problem, and that is to write

$$\text{distance apart} = \left| 198 - \frac{9t}{4} \right|$$

where the two vertical lines mean 'the **modulus** of ' or 'the **absolute value** of '.

The modulus sign causes any minus sign to be disregarded. So,

$$\text{when } t = 60, \text{ distance apart } = |\ 63\ | = 63 \text{ miles,}$$

$$\text{when } t = 120, \text{ distance apart } = |\ -72\ | = 72 \text{ miles.}$$

The mathematical way of expressing the original problem is to find t when

$$\left| 198 - \frac{9t}{4} \right| < 10 .$$

This can be solved by regarding it as short for two separate inequalities.

$$-10 < 198 - \frac{9t}{4} < 10 .$$

Writing it in this form enables two inequalities to be solved at the same time;

$$\begin{aligned} & -10 < 198 - \frac{9t}{4} < 10 \\ \Rightarrow \quad & 10 > \frac{9t}{4} - 198 > -10 \\ \Rightarrow \quad & 208 > \frac{9t}{4} > 188 \\ \Rightarrow \quad & 92\frac{4}{9} > t > 83\frac{5}{9} . \end{aligned}$$

Hence the two cars are within 10 miles of one another between $83\frac{5}{9}$ and $92\frac{4}{9}$ minutes after the start of their journeys.

Example

Solve

(a) $\left| 40\left(1 - \frac{x}{7}\right) \right| < 5$

(b) $|\ 8t - 11| \geq 13.$

Solution

(a) $\left| 40\left(1 - \frac{x}{7}\right) \right| < 5$

$$\Rightarrow \quad -5 < 40\left(1 - \frac{x}{7}\right) < 5$$

$$\Rightarrow \quad -\frac{1}{8} < 1 - \frac{x}{7} < \frac{1}{8}$$

$$\Rightarrow \quad \frac{1}{8} > \frac{x}{7} - 1 > -\frac{1}{8}$$ (multiplying by –1 changes the inequality round)

$$\Rightarrow \quad \frac{9}{8} > \frac{x}{7} > \frac{7}{8}$$

$$\Rightarrow \quad \frac{63}{8} > x > \frac{49}{8} \quad \text{or} \quad 6.125 < x < 7.875.$$

(b) $|8t - 11| \geq 13$

$$\Rightarrow \quad 8t - 11 \geq 13 \quad \text{or} \quad 8t - 11 \leq -13$$

$$\Rightarrow \quad 8t \geq 24 \quad \text{or} \quad 8t \leq -2$$

$$\Rightarrow \quad t \geq 3 \quad \text{or} \quad t \leq -\tfrac{1}{4}.$$

Example

Laila wants some leaflets printed. She gets two quotes:

> 'Bulkfast' say they will charge a fixed charge of £60 plus 5p per copy. 'Smallorder' printers quote a £25 fixed charge plus 8p per copy.

Find a formula for the difference in price if Laila decides to have *n* copies printed. She decides that a difference of £5 or less is ‘negligible’ and, if this is the case, that quality of product will be the deciding factor between the two firms. For what values of *n* is the difference ‘negligible’?

Solution

For *n* copies 'Bulkfast' will charge £$(60 + 0.05n)$ and 'Smallorder' £$(25 + 0.08n)$.

$$\text{Difference in price} = |(60 + 0.05n) - (25 + 0.08n)|$$

$$= |\ 35 - 0.03n\ |.$$

For the difference to be negligible

$$
\begin{array}{lll}
 & |\,35 - 0.03n\,| \le 5, & n \in \mathrm{N} \\
\Rightarrow & -5 \le 35 - 0.03n \le 5, & n \in \mathrm{N} \\
\Rightarrow & 5 \ge 0.03n - 35 \ge -5, & n \in \mathrm{N} \\
\Rightarrow & 40 \ge 0.03n \ge 30, & n \in \mathrm{N} \\
\Rightarrow & 1333.33 \ge n \ge 1000, & n \in \mathrm{N}.
\end{array}
$$

So the difference is negligible when n is between 1000 and 1333 copies inclusive.

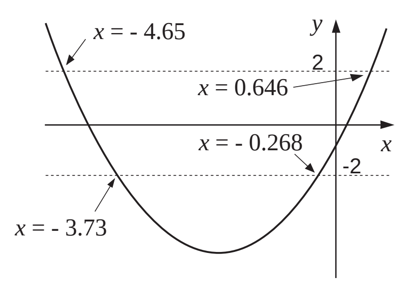
Example

Solve $\left|\, x^2 + 4x - 1 \,\right| < 2$.

Solution

$\left|\, x^2 + 4x - 1 \,\right| < 2 \Rightarrow -2 < x^2 + 4x - 1 < 2$.

This can be solved as two separate inequalities $x^2 + 4x - 3 < 0$ and $x^2 + 4x + 1 > 0$ and the results combined.



A quicker technique is to complete the square;

$$
\begin{array}{llll}
 & -2 < (x+2)^2 - 5 < 2 & & \\
\Rightarrow & 3 < (x+2)^2 < 7 & & \\
\Rightarrow & \sqrt{3} < x + 2 < \sqrt{7} & \text{or} & -\sqrt{3} > x + 2 > -\sqrt{7} \\
\Rightarrow & \sqrt{3} - 2 < x < \sqrt{7} - 2 & \text{or} & -\sqrt{3} - 2 > x > -\sqrt{7} - 2 \\
\Rightarrow & -0.268 < x < 0.646 & \text{or} & -3.73 > x > -4.65.
\end{array}
$$

Activity 19 Properties of the modulus function

Which of these statements below are true? x and y stand for any numbers.

(a) $|\,x + y\,| = |\,x\,| + |\,y\,|$ (b) $|\,x - y\,| = |\,x\,| - |\,y\,|$

(c) $|\,xy\,| = |\,x\,|\,|\,y\,|$

(d) $\left|\,\dfrac{x}{y}\,\right| = \dfrac{|\,x\,|}{|\,y\,|}$ (e) $\left(|\,x\,|\right)^2 = x^2$.

Activity 20 Graphs involving the modulus function

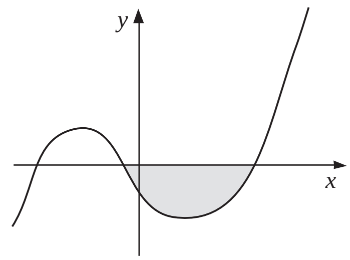
(a) Sketch the graph of $y = x$. What will the graph of $y = |x|$ look like?

(b) Sketch the graph of $y = 2x - 5$ and hence the graph of $y = |2x - 5|$

(c) Sketch the graph of $y = x^2 - 7x + 5$ and $y = \left| x^2 - 7x + 5 \right|$.

(d) The graph of $y = f(x)$ is shown on the right.

Sketch $y = \left| f(x) \right|$ and $y = f\left(|x|\right)$.



Exercise 5I

1. Solve these inequalities :

   (a) $|5a + 3| \le 2$ (b) $|10 - b| < 6$

   (c) $|15 - 4c| \ge 3$ (d) $\left| \frac{2d}{3} + 7 \right| \le 11$

   (e) $\left| 2(5 + \frac{12e}{5}) \right| < 16$ (f) $\left| \frac{4 - 9f}{13} \right| \le 10$

2. Use a graph to solve :

   (a) $\left| 12x^2 + 7x - 10 \right| < 20, x \in \mathbb{N}$

   (b) $\left| 100 - 5x - x^2 \right| < 50, x \in \mathbb{N}$

3. Solve

   (a) $\left| x^2 - 8x + 10 \right| < 3$ (b) $\left| x^2 + 2x - 6 \right| > 6$

   (c) $\left| x^2 + 10x + 18 \right| < 8$.

4. The formula $F = \frac{9}{5}C + 32$ is used to convert degrees Celsius (C) into degrees Fahrenheit (F). Often, however, the formula $F = 2C + 30$ is used to make the arithmetic easier. Over what temperature range (in °C) is the approximate formula correct

   (a) to within 5°F (b) to within 2°F.

5. Di and Vi decide to race each other over 100 metres. Vi is a faster sprinter than Di. Vi's average time for the 100 metres is 12.5 seconds whilst Di's is 16 seconds. It is agreed that, while Vi will run the full distance, Di will be given a head start of d metres.

   (a) Suppose they run the race at the same speed as their average times suggest. How many seconds apart will they cross the finishing line?

   (b) They decide to set d so that no more than 1 second is expected between them at the finish. Find over what values d could range.

5.10 Miscellaneous Exercises

1. In setting an exam paper, it is agreed that short questions will carry 4 marks each and long questions 13 marks. The paper must contain 16 questions altogether and must be out of 100 marks in total.

   How many of the two types of question must there be?

   (You should assume that candidates have to answer all the questions.)

2. This question is about taxation. At the time of writing tax is calculated like this:

   - first £3005 earned in a year is not taxed.
   - income between £3005 and £20 700 is taxed at 25%;
   - all income over £20 700 is taxed at 40%.

   Two brothers compare their 1990/91 tax bills. Peter notices that, although his income was twice as much as Jonathan's, he paid 2.8 times as much tax. How much money did Peter earn? (Assume that neither brother receives any allowance other than the £3005 mentioned above.)

3. Staff at a factory are working overtime to get a job finished quickly. It is decided that continuous supervision is needed. Six supervisors are chosen, each to work a 6-hour shift, with an equal overlap time between each shift. The job will not take less than 32 hours and must be finished within 35 hours. How long could the overlap between shifts be?

4. (a) Clair needs to get to work by 9.00 am. She finds that if she leaves home at 7.00 am, the journey takes 1 hour, but that every two minutes after 7 o'clock adds another minute to her travelling time. At what time must she leave home in order to get to work exactly on time?

   (b) Her journey home takes 1 hour if she leaves at 4.30 pm. Every 4 minutes later than this adds 3 minutes to her travelling time. To the nearest minute, when must she leave work to arrive home at 6.00 pm?

5. Economists often talk about a firm's total cost function. This function relates the total cost C to the level of output Q units.

   Suppose a firm's total cost function is

   $$C = 4Q^2 + 100Q + 16000,$$

   find the values of $Q(\in \mathbb{N})$ for the total cost to be less than £70 000.

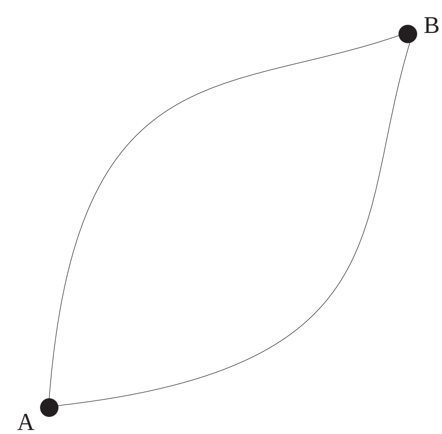
6. An office manager employs the following method of buying Christmas presents for staff. He buys a consignment of turkeys, all at the same price. 15 of them are reserved for his colleagues and the rest are resold when the price has risen by £2 per turkey above what he paid. He then sells the reserved turkeys at whatever price he needs to break even.

   This year he had to pay £1200 for the original consignment, and his colleagues paid exactly half the original price for their birds. How many turkeys did he buy?

7. A light plane travels from London to Inverness, a distance of 450 miles, at an average speed of 200 mph. Another plane travels the same route but gets there half an hour quicker. How fast was the second plane travelling?

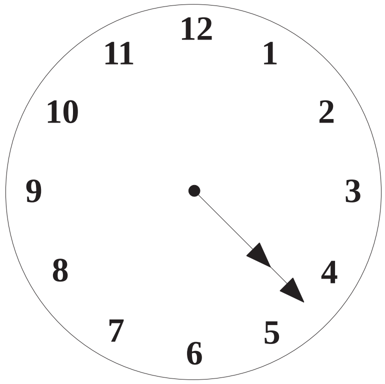
8. Two cars travel 150 miles along the motorway. On average, one car travels 10 mph faster than the other, and completes the journey 15 minutes before the other one. What speed was the slower one doing, to the nearest mile per hour?

9. In a cycle race the contestants ride from A to B and back. Both outward and return journeys are the same distance, but the return journey is mostly downhill; average speeds on the return leg are 8 km/h faster than on the outward.

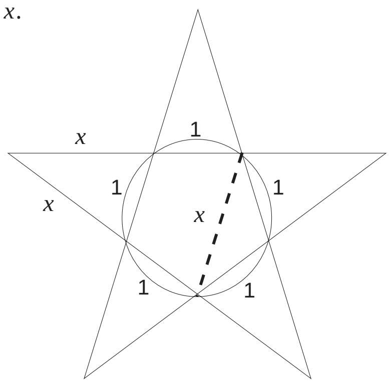
   Neville covers the 60 km course in $2\frac{1}{2}$ hours. Work out his average speed for the first half of the course, to the nearest km/h.

10.

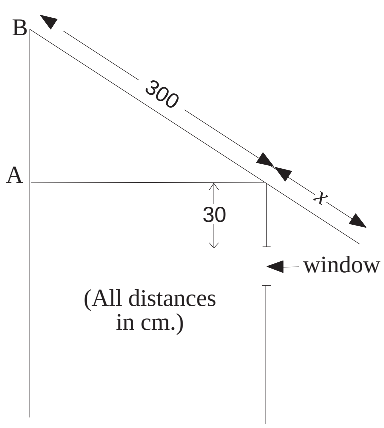


The hands on the clock move smoothly and continually. In the above picture the minute hand is exactly covering the hour hand. What time is it? (NOT 4.20)

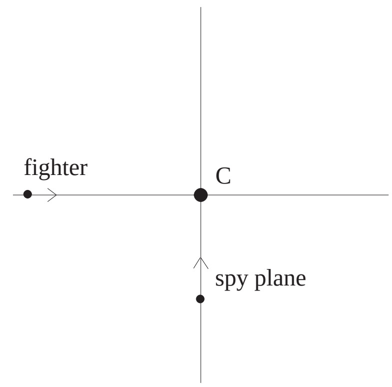
11. The shape below is known as a pentagram. Find *x*.



12. The diagram shows the elevation of a garden shed, the roof of which overhangs to the level of the top of the window. The distance AB is 30 cm longer than the length of the overhang, marked *x* in the diagram. Find *x*.



13. A fighter and a spy plane are travelling on perpendicular paths that cross at the point C. At 1.10 pm the fighter is 100 miles due west of C and travelling east at 400 mph. At the same time the spy plane is 60 miles due south of C and travelling north at 300 mph.



(a) If *t* is the time in hours after 1.10 pm, write down formulae for the distances of both planes from C at time *t*.

(b) For how many minutes are the two planes within 50 miles of each other?

14. A rectangular paddock is 3 times longer than it is wide. The width is increased by 20 m. This has the effect of doubling the area. Find the original dimensions of the paddock.

15. Solve these:

(a) $|x| = |x+1|$ (b) $\dfrac{12}{x+1} > \dfrac{x}{6}$

(c) $x^4 - 13x^2 + 36 = 0$ (d) $x - 6\sqrt{x} + 8 = 0$

(e) $x^3 + 2x^2 - 48x = 0$ (f) $x^2 - |x| - 2 = 0$.