

3 FUNCTIONS

Objectives

After studying this chapter you should

- understand what is meant by a composite function;
- understand the difference between $f(g(x))$ and $g(f(x))$;
- know what is meant by the inverse of a function;
- be able to sketch the graph of a function's inverse.

3.0 Introduction

You have now met a number of functions. Some have represented practical relationships whilst others have been simply mathematical functions. In this chapter, you will extend these ideas by looking at how two functions can be used to define another function, and considering how to find inverse functions, and what they represent. Whilst many of the functions to be studied will be quite complex mathematically, it is the practical application of mathematics which motivates the need for those extensions.

3.1 Composite functions

You have already met some quite complicated functions in the first two chapters. In this section you will see how **composite** functions can be built up, and why they are an important concept in mathematics. The idea of a composite function is introduced with a practical currency exchange rate example.

Example

A bank in the UK offers the exchange rate

$$£1 = \$1.7$$

plus an administration payment of £2 for each transaction. A similar shop in USA, offers the exchange rate

$$\$1 = 1.6 \text{ Swiss francs}$$

plus an administration payment of \$3 for each transaction. How many Swiss francs will you actually receive if you first exchange £10 into dollars in the UK, and then into Swiss francs in the USA?

Solution

It is easy enough to solve this problem in two numerical stages. Firstly the £10 is changed to dollars. Taking away the £2 transaction payment, leaves you with $\$1.7 \times 8 = \13.6 .

Secondly the \$13.6 are changed into Swiss francs, remembering to take off \$3 transaction payment. So you have

$$10.6 \times 1.6 \text{ Swiss francs} = 16.96 \text{ Swiss francs.}$$

A more general approach is to form two functions to represent the two transactions. If $t =$ amount in £, $x =$ amount in \$ and $y =$ amount in Swiss francs, then for the first transaction

$$\begin{aligned} x &= 1.7(t - 2) \\ &= 1.7t - 3.4 \end{aligned}$$

and, since x is a function of t , write

$$x(t) = 1.7t - 3.4. \quad (1)$$

Similarly, for the second transaction

$$\begin{aligned} y &= 1.6(x - 3) \\ &= 1.6x - 4.8 \end{aligned}$$

and write

$$y(x) = 1.6x - 4.8 \quad (2)$$

to show that y is a function of x . Since x in turn is a function of t , you can write y as a function of t by substituting (1) in (2) to give

$$\begin{aligned} y(x) &= y(x(t)) \\ &= 1.6x(t) - 4.8 \\ &= 1.6(1.7t - 3.4) - 4.8 \\ &= 2.72t - 5.44 - 4.8 \\ &= 2.72t - 10.24 \end{aligned}$$

i.e. $f(t) = 2.72t - 10.24 \quad (3)$

where $f(t) = y(x(t))$.

Does equation (3) give the correct solution when $t = 10$?

The composite function $y(x(t))$ is written for short as $y \circ x$.

Example

The functions f and g are defined by

$$f(t) = 4t - 3$$

$$g(t) = 2t - 1$$

Find the composite functions $f \circ g$ and $g \circ f$.

Solution

$$\begin{aligned} f \circ g &= f(g(t)) = f(2t - 1) \\ &= 4(2t - 1) - 3 \\ &= 8t - 7. \end{aligned}$$

whereas

$$\begin{aligned} g \circ f &= g(f(t)) = g(4t - 3) \\ &= 2(4t - 3) - 1 \\ &= 8t - 7. \end{aligned}$$

In this case the two composite functions $f \circ g$ and $g \circ f$ are identical. This is not generally true as you will see in the activity below.

Activity 1

The function f and g are defined by

$$f(x) = x^2 + 1$$

$$g(x) = x - 1$$

Is $f \circ g = g \circ f$ for any value of x ?

As you should have seen composite functions are **not** usually commutative. That is, in general $f \circ g \neq g \circ f$. For instance, if

$$f(x) = x + 3, x \in \mathbb{R}$$

and $g(x) = 2x, x \in \mathbb{R}$

then
$$f(g(x)) = f(2x)$$

$$= 2x + 3 \text{ for } x \in \mathbb{R}.$$

Similarly
$$g(f(x)) = g(x + 3)$$

$$= 2(x + 3)$$

$$= 2x + 6 \text{ for } x \in \mathbb{R}.$$

The two composite functions are clearly different for all values of x . Also note that, because the range of the function which is applied first is the domain for the second function, it is essential that the range of the first is suitable as a domain for the second.

Example

If $h(x) = \frac{1}{x}$, $x \in \mathbb{R}$, $x \neq 0$ and $m(x) = x - 2$, $x \in \mathbb{R}$, find suitable domains for the composite function $m \circ h$ and $h \circ m$.

Solution

Although $m(h(x)) = \frac{1}{x} - 2$, $x \in \mathbb{R}$, $x \neq 0$ is a function, there is a problem with the composite function $h(m(x))$.

$$h(m(x)) = h(x - 2)$$

$$= \frac{1}{x - 2}.$$

If the domain is chosen to be $x \in \mathbb{R}$, $x \neq 0$ which is the domain of the first function m , then x can be equal to 2.

However, $h(m(2)) = \frac{1}{2-2} = \frac{1}{0}$, and this does not exist.

So $x \in \mathbb{R}$, $x \neq 0$ is not suitable as a domain for $h \circ m$.

If $x \in \mathbb{R}$, $x \neq 2$ is chosen as the domain, $h \circ m$ will be defined for all the values given in the domain.

Exercise 3A

- Work out composite functions $f \circ g$ and $g \circ f$ for each of the following pairs of functions. In each case, state a suitable domain for the composite function. (You may find a sketch of the composite function graph made with a computer or calculator helpful when checking your answer.)
 - $f(x) = x - 1$ and $g(x) = x^3$
 - $f(x) = +\sqrt{x}$ and $g(x) = x - 2$
 - $f(x) = \frac{1}{x}$ and $g(x) = x + 1$
 - $f(x) = x^2 - 1$ and $g(x) = \frac{1}{x}$
 - $f(x) = x + 3$ and $g(x) = x - 3$
 - $f(x) = 6 - x$ and $g(x) = 6 - x$.
- Find the composite function $f \circ g$ and $g \circ f$ if $f(x) = 1 + \frac{1}{x}$ and $g(x) = x^2$.
- Find the composite function $h \circ f \circ g$ if $f(x) = x - 3$, $g(x) = x^2$ and $h(x) = \frac{1}{x}$.

3.2 Inverse functions

In Chapter 2 you met the function that transforms degrees Celsius to degrees Fahrenheit, namely

$$F = \frac{9}{5}C + 32. \quad (1)$$

Suppose you wanted to find the formula that gives degrees Celsius in terms of degrees Fahrenheit, then taking 32 from both sides

$$F - 32 = \frac{9}{5}C$$

and multiplying by $\frac{5}{9}$ gives

$$C = \frac{5}{9}(F - 32) \quad (2)$$

This is an example of an **inverse** function.

Example

If y = number of dollars and x = equivalent number of pounds and $y = 1.7x$, express x in terms of y .

Solution

You must make x the subject of the formula when $y = 1.7x$.

This gives $x = \frac{y}{1.7}$

(in fact, exchange rates from dollars to pounds and pounds to dollars are not in practice equivalent, and there is usually a transaction charge)

Activity 2 Kepler's third law

In 1619 the astronomer *Kepler* announced his third law of planetary motion (dedicated to James II of England) which stated that the periodic time of a planet, T , is related to its average radius of orbit, say R , by the formula

$$T = kR^{\frac{3}{2}}.$$

Find the inverse function which expresses R as a function of T .

A special notation is introduced for inverse functions. For example, the temperature conversion formula, with x now denoting degrees Celsius and y degrees Fahrenheit,

$$y = f(x) = \frac{9}{5}x + 32 \quad (3)$$

can be rearranged to give, $y - 32 = \frac{9}{5}x$ and $x = \frac{5}{9}(y - 32)$.

The inverse function is denoted by f^{-1} , so

$$f^{-1}(y) = \frac{5}{9}(y - 32).$$

Since y could be any variable, we can rewrite the inverse function as a function of the variable x as

$$f^{-1}(x) = \frac{5}{9}(x - 32). \quad (4)$$

Note that the meaning of the variable x is different in (3) and (4). In (3), it represents the temperature in degrees Celsius, so that, for example, 20°C will transform to $\frac{9}{5} \times 20 + 32 = 36 + 32 = 68^\circ\text{F}$. Whilst in (4), x represents degrees Fahrenheit, so that 77°F transforms to $\frac{5}{9} \times (77 - 32) = \frac{5}{9} \times 45 = 25^\circ\text{C}$.

Example

Find the inverse of $f(x) = \frac{1}{1-x} + 2$, $x \in \mathbb{R}$, $x \neq 1$, and state the domain of the inverse function.

Solution

Let $f(x) = y$, so that

$$y = \frac{1}{1-x} + 2$$

$$\Rightarrow y - 2 = \frac{1}{1-x}$$

$$\Rightarrow (1-x)(y-2) = 1$$

$$\Rightarrow 1-x = \frac{1}{y-2}$$

$$\Rightarrow x = 1 - \frac{1}{y-2}.$$

This formula gives the inverse function as

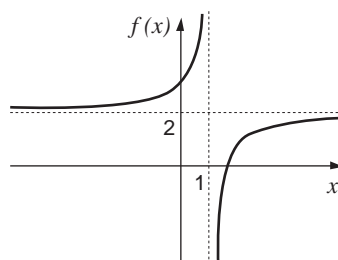
$$f^{-1}(y) = 1 - \frac{1}{y-2}.$$

Replacing y by x , this becomes

$$f^{-1}(x) = 1 - \frac{1}{x-2}.$$

As $f^{-1}(x)$ is the inverse of $f(x)$, its **domain** will be the **range** of $f(x)$. This is because its task is to map members of the original range back onto the corresponding members of the domain.

The figure opposite shows the graph of $f(x)$. The range is all the real numbers, except 2. There is no value of x for which $f(x) = 2$, as shown by the horizontal dotted line.

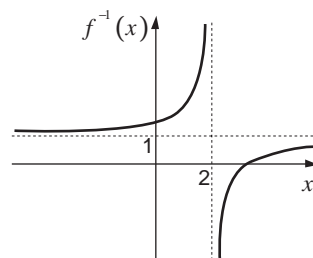


So the domain of $f^{-1}(x)$ is the set of real numbers, except 2. This means the full definition of $f^{-1}(x)$ is

$$f^{-1}(x) = 1 - \frac{1}{x-2}, \quad x \in \mathbb{R}, \quad x \neq 2.$$

The graph of $f^{-1}(x)$ is shown opposite.

(Note also that the range of $f^{-1}(x)$ is the domain of $f(x)$).



Exercise 3B

Find the inverses of these functions.

State the domain of each inverse.

1. $f(x) = x + 2, x \in \mathbb{R}$

2. $f(x) = 4x - 1, x \in \mathbb{R}$

3. $f(x) = 4x - 2, x \in \mathbb{R}$

4. $f(x) = x, x \in \mathbb{R}$

5. $f(x) = 1 - x, x \in \mathbb{R}$

6. $f(x) = \frac{3}{x}, x \in \mathbb{R}, x \neq 0$

7. $f(x) = \frac{1}{x+2}, x \in \mathbb{R}, x \neq -2$

8. $f(x) = \frac{1}{5-x} + 2, x \in \mathbb{R}, x \neq 5.$

3.3 Symmetry about the line $y = x$

Functions and their inverses have an interesting geometrical property as you will see below.

Activity 3 The graph of an inverse function

Below is a list of functions, each with its inverse. On graph paper, plot and draw the graph of a function, together with its inverse, on the same axes. Repeat this process for each function in the list. Use the same scale on both axes. Use values of x between -6 and $+6$, and y values between -12 and $+12$.

(a) $f(x) = 2x, f^{-1}(x) = \frac{1}{2}x$;

(b) $f(x) = x - 4, f^{-1}(x) = x + 4$;

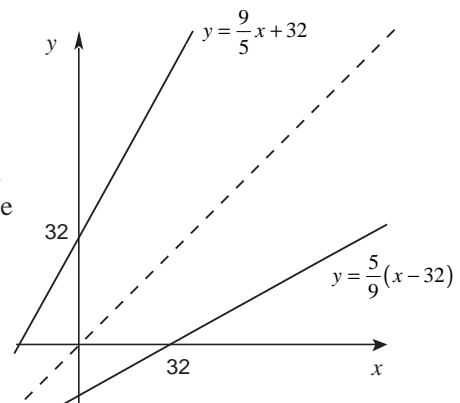
(c) $f(x) = x^2, (x \in \mathbb{R}, x \geq 0), f^{-1}(x) = +\sqrt{x}, (x \in \mathbb{R}, x \geq 0).$

Describe the relationship between the graph of each function and that of its inverse. You may find drawing the line $y = x$ on each pair of axes helpful.

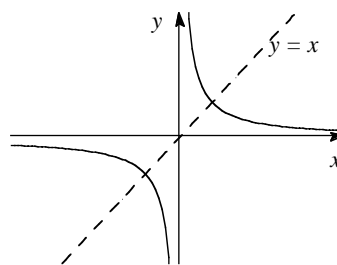
In the previous Section 3.2, the inverse of the function

$$f(x) = \frac{9}{5}x + 32 \text{ was found to be } f^{-1}(x) = \frac{5}{9}(x - 32).$$

The figure opposite shows the graphs of these two functions on the same pair of axes. The dotted line is the graph $y = x$. These graphs illustrate a general relationship between the graph of a function and that of its inverse, namely that one graph is the **reflection** of the other in the line $y = x$.



There is an interesting group of functions which have graphs that are symmetrical about the line $y = x$. The figure opposite shows such a graph, for the function $f(x) = \frac{1}{x}$. The graph of its inverse function will be the reflection of this curve in the dotted line, but as the curve is symmetrical about this line, the inverse function will be the same as the function itself.



A function which is its own inverse is called a **self inverse** function. In this case, if $f(x) = \frac{1}{x}$, then $f^{-1}(x) = \frac{1}{x}$ too.

One final point that is worth noting is that

$$f(f^{-1}(x)) = f^{-1}(f(x)) = x.$$

For example, you have already seen that if $f(x) = \frac{9}{5}x + 32$,

then
$$f^{-1}(x) = \frac{5}{9}(x - 32).$$

Now
$$\begin{aligned} f(f^{-1}(x)) &= f\left(\frac{5}{9}(x - 32)\right) \\ &= \frac{9}{5}\left(\frac{5}{9}(x - 32)\right) + 32 \\ &= x - 32 + 32 \\ &= x, \end{aligned}$$

and similarly for $f^{-1}(f(x))$.

Activity 4

If $f(x) = 4x - 3$, find $f^{-1}(x)$ and check that

$$f(f^{-1}(x)) = f^{-1}(f(x)) = x.$$

Exercise 3C

1. Use a graphic calculator or computer to make sketches of each of these functions. (You can plot and draw them if you do not have a calculator or computer at hand). Use values of x and y between -5 and $+5$ and on each pair of axes show the graph of the inverse function. You will find that superimposing the line $y = x$ is helpful. Use the same scales for x and y .

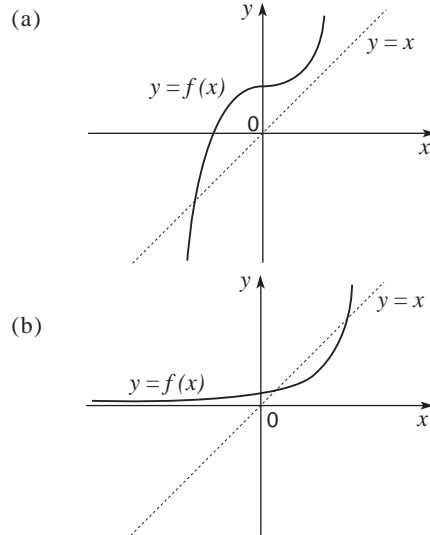
(a) $f(x) = x + 3$ (b) $f(x) = x^3$ (c) $f(x) = -2x$

(d) $f(x) = \frac{-1}{x}$ (e) $f(x) = 4 - x$ (f) $f(x) = \frac{1}{x^3}$.

2. For (b) and (c) in Question 1 above check that

$$f(f^{-1}(x)) = f^{-1}(f(x)) = x.$$

3. Copy the graphs below and sketch the graphs of the inverse functions.

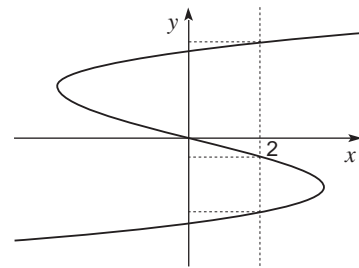


3.4 Functions with no inverse

In Section 2.1 you saw that for an algebraic rule, or formula, to be a function, the rule must map each member of the domain to one and only one member of the range. That is, it must give only one answer. So, for example, if $y^2 = x$ then it is not possible to express y as a function of x since $y = \pm\sqrt{x}$; that is, for $x = 4$, $y = \pm 2$, etc. One value of x gives two values of y .

However, $y = \sqrt{x}$, $x \geq 0$, is a function, as there is only one number that is the positive square root of x , for any real number x , which is greater than zero.

You can tell if a mapping is a function from its graph. If a line parallel to the y -axis crosses the graph in more than one place, the mapping is **not** a function. For instance, the figure opposite shows that the value $x=2$ in the domain is mapped to three values in the range, so this is not the graph of a function.



A **many to one** function will map different members of its domain to the same member of the range. An example of such a function is

$$f(x) = x^2, x \in \mathbb{R}.$$

For instance $f(3) = 3^2 = 9$ and $f(-3) = (-3)^2 = 9$. So two values in the domain, 3 and -3 , are mapped to the same value in the range, namely 9 .

Activity 5

Use a calculator or computer to draw the graph of $y = x^2$ for values of x between -3 and $+3$. Use the same scales on both axes. Draw on the same pair of axes the graph of the inverse mapping, by using the reflection property.

Does the graph represent a function?

If the domain of $f(x)$ is changed from $x \in \mathbb{R}$ to $x \in \mathbb{R}, x \geq 0$, make a sketch of $f(x)$ on new axes. Also show what the graph of the inverse must look like on the same pair of axes. Is this the graph of a function?

Activity 6

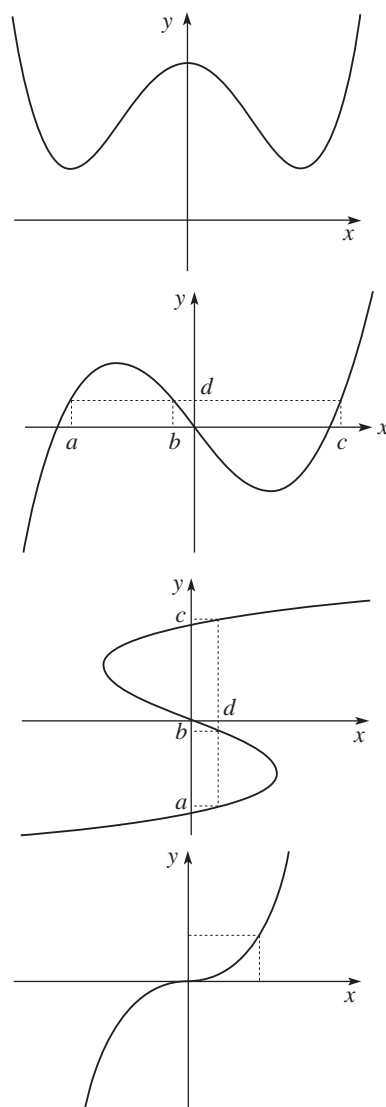
An even function has a graph which is symmetrical about the y -axis. Draw some sketches of functions which are even, like the one shown opposite. Then show on the same axes what the inverse mapping should look like.

A many to one function maps different values in the domain to the same value in the range. The graph of a many to one function can be crossed in more than one place by a line parallel to the x -axis, as shown opposite.

The inverse of such a function must therefore map the same value to different members of the original function's domain. In the graph opposite, d must be mapped to a, b and c . This means that the inverse **cannot** be a function, (as it does not send every member of **its** domain to only one member of its range).

So a many to one function cannot have an inverse function.

If a function is to have an inverse, the function cannot be many to one - it must be **one to one**. That is, every member of its domain is mapped to its own unique member of the range. The graph of such a function can only be crossed by a line parallel to the x -axis **once**, as shown in the figure opposite.



Exercise 3D

Use a graphic calculator or computer to sketch the graphs of these functions. Check whether a horizontal line can be drawn which crosses each graph in more than one place to decide whether each graph is one to one or many to one. For the functions which are one to one draw the graph of the inverse function in each case using the reflection relationship.

1. $f(x) = x^2 + 2, x \in \mathbb{R}$

2. $f(x) = x^3 + 2, x \in \mathbb{R}$

3. $f(x) = \frac{1}{x^2}, x \in \mathbb{R}, x \neq 0$

4. $f(x) = (x+1)^2, x \in \mathbb{R}$

5. $f(x) = x^3 + x^2 - 6x, x \in \mathbb{R}$

6. $f(x) = x^3 - 9x^2 + 27x - 27, x \in \mathbb{R}$

7. $f(x) = \frac{1}{x-2} + 1, x \in \mathbb{R}, x \neq 2$

8. $f(x) = 3, x \in \mathbb{R}$.

3.5 Modelling repeating patterns

The sound produced by a tuning fork is a wave shape when its 'loudness' is plotted against time. This is shown in the first graph opposite. This wave repeats itself after 0.02 seconds - the **period** of the wave.

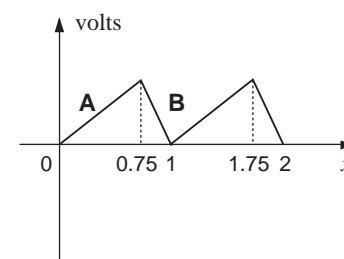
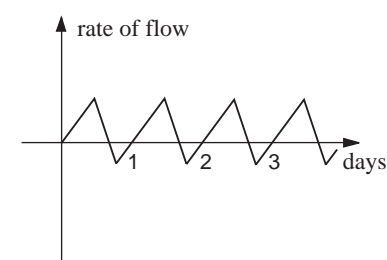
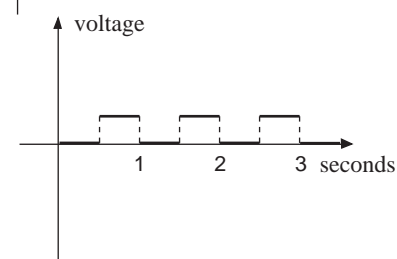
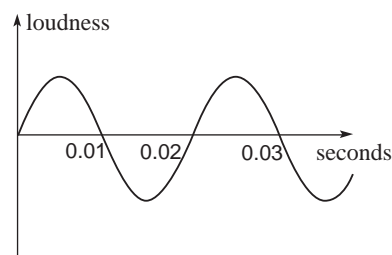
Electronic clocks count **beats** produced by an internal circuit. Some circuits, for instance, can be made to produce a voltage which repeats itself as shown in the second graph. The period here is 1 second, as the pattern is repeated after this time. These are called square waves.

A number of living organisms, especially plants, exhibit repeating cycles of behaviour. For instance, a plant's stem may conduct sap to its leaves in a daily repeating pattern, as shown in the third graph.

Patterns like these are very common. The most important among them are based on trigonometric functions, and these are covered separately in Chapter 10. However, some can be modelled with the functions already covered.

Example

A 'sawtooth' oscillation is a pattern which occurs in electronics. An engineer requires such an oscillation to have the shape shown opposite, repeating once every second.



The equation of the line marked A is $y = \frac{8x}{3}$. The equation of the line marked B is $y = -8x + 8$. So the function which models

$$\text{the first 'tooth' is } f(x) = \begin{cases} \frac{8}{3}x, & 0 \leq x \leq 0.75 \\ -8x + 8, & 0.75 \leq x \leq 1 \end{cases}$$

The period of the function is 1 second, and this completes the definition of the function.

Example

The rate of flow of sap in a certain species of plant is thought to follow the pattern as shown opposite. A positive rate of flow means the flow is upwards.

The section of the graph for values of x between 0 and 12 is the graph of

$$y = \frac{220}{144}x^2 - 100.$$

The part when x is between 12 and 18 is constant at 120, so its equation for these values of x is $y = 120$. The last part of the cycle, when x is between 18 and 24, is given by the graph of

$$y = -\frac{220}{6}x + 780.$$

So the function which models this pattern is

$$f(x) = \begin{cases} \frac{220}{144}x^2 - 100 & \text{for } 0 \leq x \leq 12 \\ 120 & \text{for } 12 \leq x \leq 18 \\ -\frac{220}{6}x + 780 & \text{for } 18 \leq x \leq 24 \end{cases}$$

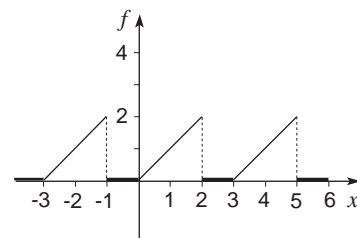
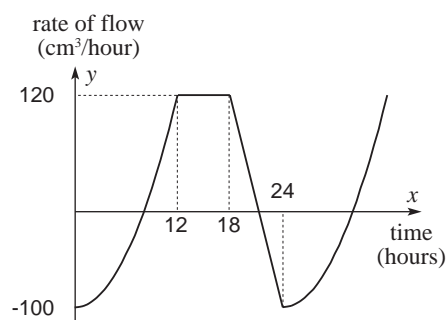
with a period of 24 hours.

Example

Sketch the graph $f(x) = \begin{cases} x & \text{for } 0 \leq x < 2 \\ 0 & \text{for } 2 \leq x \leq 3 \end{cases}$

when $f(x)$ has a period of 3 units, for values of x between -3 and 6.

The solution is shown opposite.



Exercise 3E

1. A square wave generator produces a voltage that is given by the function $f(x)$, where

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} < x \leq 1 \end{cases}$$

and $f(x)$ has a period of 1 unit. Draw the graph for values of x between 0 and 3.

2. The stock level of coal at a small power station (in thousand tonne units) can be modelled by the function

$$f(x) = 20 - 2x \text{ for } 0 \leq x \leq 7$$

where x is the number of days. The function has a period of seven days. Draw the graph of the function for values of x between 0 and 21. What is the lowest level the stocks ever reach, and how regular are the deliveries? Why would the following function be unlikely to represent a stock level problem:

$$g(x) = 20 - 2x \text{ for } 0 \leq x \leq 10?$$

3. Draw the graph of $f(x)$ for values of x between 0 and 12, where

$$f(x) = \begin{cases} 1 - (x-1)^2 & \text{for } 0 \leq x \leq 2 \\ (x-3)^2 - 1 & \text{for } 2 \leq x \leq 4 \end{cases}$$

$f(x)$ has a period of 4 units. You may need to plot and draw the first 'cycle' of the pattern.

4. Plot and draw the graph of the function below, showing a full two periods of its cycle.

$f(x)$ has a period of 2 units. You may need to plot and draw the first 'cycle' of the pattern.

$$f(x) = \begin{cases} \frac{1}{2}x^2 & \text{for } 0 \leq x \leq 1 \\ \frac{1}{x} - \frac{1}{2} & \text{for } 1 \leq x \leq 2 \end{cases}$$

3.6 Miscellaneous Exercises

1. Using a graphic calculator or computer to make sketches of the following functions decide whether each has an inverse function or not. If an inverse exists, find the algebraic rule for it, and state its domain.

(a) $f(x) = 3x - 2, x \in \mathbb{R}$

(b) $f(x) = 4 - 3x, x \in \mathbb{R}$

(c) $f(x) = x^2 - 1, x \in \mathbb{R}$

(d) $f(x) = x^2 - 1, x \in \mathbb{R}, x \geq 0$

(e) $f(x) = (x-3)^2, x \in \mathbb{R}$

(f) $f(x) = \frac{1}{x} - 4, x \in \mathbb{R}, x \neq 0$

(g) $f(x) = \frac{1}{(x+1)^2 - 2x}, x \in \mathbb{R}, x \neq \pm 1.$

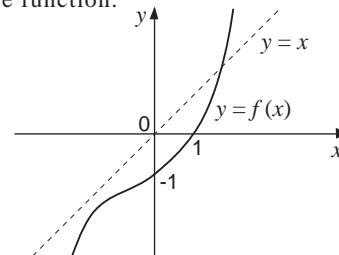
2. Use a graphic calculator to sketch the following functions. Use the sketch to superimpose the graph of the inverse function in each case.

(a) $f(x) = x^2 + 4, x \in \mathbb{R}, x > 0$

(b) $f(x) = \frac{1}{x-2} + 1, x \in \mathbb{R}, x \neq 2$

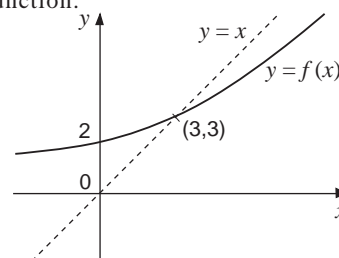
(c) $f(x) = 4x - 1, x \in \mathbb{R}.$

3. Copy the graph and superimpose the graph of the inverse function.



Show where the graph of $f^{-1}(x)$ cuts the x and y axes.

4. Copy the graph below, and superimpose the graph of $f(x-2)$. Hence sketch the inverse of this function.



5. Plot and draw the following periodic functions, showing two full periods for each.

$$(a) f(x) = \begin{cases} x & \text{for } 0 \leq x \leq 2 \\ 4-x & \text{for } 2 \leq x \leq 4 \end{cases}$$

$$(b) f(x) = \begin{cases} x^2 & \text{for } 0 \leq x \leq 1 \\ 1 & \text{for } 1 \leq x \leq 2 \end{cases}$$

$$(c) f(x) = 2 - \frac{x}{2} \text{ for } 0 \leq x \leq 2$$

$$(d) f(x) = x^5 \text{ for } 0 \leq x \leq 1.$$

