2 USING GRAPHS

Objectives

After studying this chapter you should

- be able to illustrate simple functions with a graph;
- understand what is meant by mapping, domains and ranges;
- be able to identify if a function is odd or even or neither.

2.0 Introduction

Graphs can be used to quickly get an idea of how one quantity varies as another quantity changes. This can be very useful when trying to solve a wide range of problems. An illustration is given in the first activity which deals with currency exchange.

Activity 1 Conversion rates

Conversion rates between different currencies are often displayed in banks and travel agents. You may either assume the rates given below, or find out the current rates, to draw 3 graphs showing the number of kroner, dollars and lire you can buy for any number of pounds sterling up to £50.

£1 = Danish kroner	£1 = $1.65	£1 = 2.90 Turkish lire

Use your graphs to do these conversions.

(a) £25 into Danish kroner,

(b) $35 into pounds,

(c) 80 Turkish lire into Danish kroner.

Banks usually charge a fixed commission of about £2 every time they change currency for you. On the same three pairs of axes you drew earlier, draw conversion graphs which take this commission into account. What are the values of the conversions given above now?

Another problem which can be readily illustrated graphically is that of temperature conversion. You are probably familiar with the rule for conversion from degrees Celsius (°C) to degrees Fahrenheit (°F). You multiply by 9, divide by 5 and add on 32.

This can be written as a mathematical formula

$$F = \frac{9}{5} \times C + 32.$$

Instead of using algebra you can draw a graph of F against C and use it to convert from degrees Celsius to degrees Fahrenheit.

Activity 2 Temperature conversions

Since $32°\text{F} = 0°\text{C}$ and $212°\text{F} = 100°\text{C}$, plot these two points on a graph and draw a straight line to join them. Use your graph to convert

(a) $30°\text{C}$ to $°\text{F}$ (b) $10°\text{C}$ to $°\text{F}$ (c) $100°\text{F}$ to $°\text{C}$.

The lowest possible temperature is $-273°\text{C}$. Can you use your graph to find the corresponding temperature in Fahrenheit?

2.1 Mappings, domains and ranges

Very often when trying to solve a problem you may produce a rule which links one quantity with another. For instance, the speed of a car may be linked with its braking distance, or two currencies may be linked by their rate of exchange. Once a rule or formula has been produced, it is tempting to draw a graph illustrating it, to help solve the problem. However, not all of the graph may be relevant.

Activity 3 What happens at $x = 0$?

Use a graph plotting program or calculator to draw the graphs of

$$y = \frac{1}{x} \text{ and } y = \sqrt{x}.$$

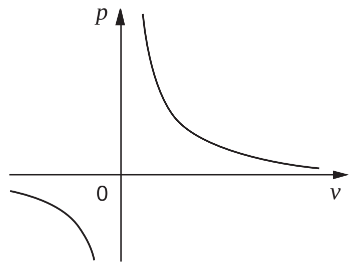
What happens to the first graph when $x = 0$? Why do you think this happens?

Why do you think there is no graph when $x < 0$ for the second equation?

Another example is given by the relationship between pressure and volume. For a fixed amount of any gas, kept at a fixed temperature, pressure and volume are linked by the formula

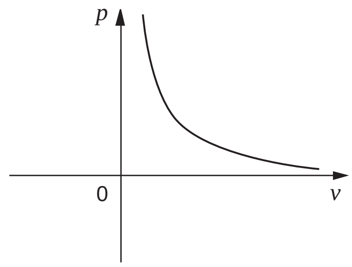
$$p = \frac{k}{v},$$

where p is the pressure, v is the volume and k is a constant.



The graph for this formula is shown opposite. It shows that the pressure of the gas gets higher as the volume gets smaller in the right half of the graph. Then the pressure 'jumps', so that it is negative as soon as the volume is negative. This does not make sense. A gas cannot have a negative volume, so the left 'branch' does not exist. To show that $p = \frac{k}{v}$ can only be used for positive values of v, the formula can be written as

$$\boxed{p = \frac{k}{v},\ v > 0}$$



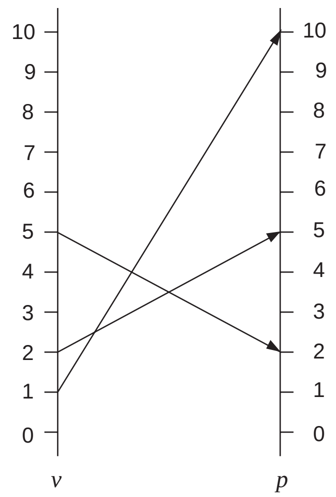
The figure opposite shows the graph of this rule, which now makes sense. As the volume increases, the pressure gets closer to zero. As the volume gets closer to zero, so the pressure gets higher - but the volume never actually equals zero.

Mapping diagrams

The equation $p = \frac{k}{v}$ can be thought of as a way of linking a value of the volume, v, with a value of the pressure. For instance, if $k = 10$, then $p = \frac{10}{v}$. So if $v = 1$, $p = \frac{10}{1} = 10$.

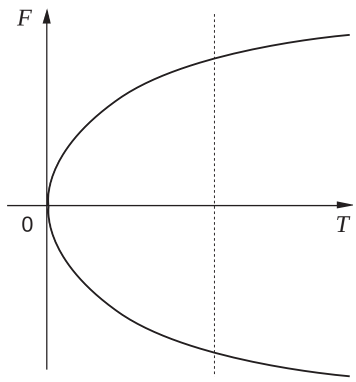
Thus 1 is 'sent' or 'mapped' to 10 by the equation.

If $v = 2$, $p = \frac{10}{2} = 5$. So 2 is 'mapped' to 5.



The figure opposite shows how several possible values of v are mapped to corresponding values of p. The formula $p = \frac{k}{v}$ is a **mapping -** it links the values in one set of numbers (here the set of volumes) with another set of numbers (the set of possible pressures). The set of all possible volumes, greater than zero, is called the **domain** of the mapping. The set of all possible pressures (again, any number greater than zero) is the **range** of the mapping.

Another example of a possible mapping is $F = \sqrt{T},\ T \geq 0$. The domain is any number greater than or equal to zero. This mapping is rather different to $p = \frac{k}{v}$, however, as it gives two 'answers' for every value in the domain. For instance, $\sqrt{4}$ is $+2$ or -2, since $(+2)^2 = 4$ and $(-2)^2 = 4$. The figure opposite shows the graph of F. The range of F is any number, positive or negative.



A mapping which gives one, and only one value, for every number in its domain is called a **function**. So the mapping

$$p = \frac{k}{v},\ v > 0$$

is a function, but the mapping

$$F = \sqrt{T},\ T \geq 0$$

is not, as it gives more than one answer for some members of its domain.

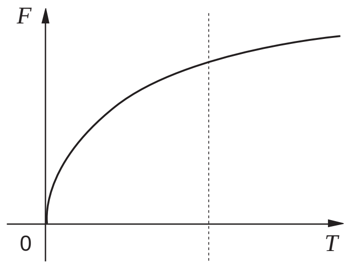
Changing the rule to get a function

Notice that the domain can be vital when deciding if a mapping is a function or not. For example, if the domain of $p = \frac{k}{v}$ is any value of v, positive or negative, including zero, the mapping ceases to be a function. This is because when $v = 0$, p cannot be calculated.

However, it is possible to adapt the rule for F to make it into a function. Suppose

$$F = +\sqrt{T},\ T \geq 0$$

which means that F is the positive square root of T. The graph of this mapping is shown opposite. If a vertical line is drawn across the graph, it will cross the graph only once. (In the previous figure a vertical line will cross the graph only twice - showing that there are two members of the range for each member of the domain). In fact mathematicians avoid this problem by usually agreeing that $\sqrt{x}$ means the positive square root.



Number sets

Sets of numbers which are often used as domains or ranges have names, so that they can be described accurately. These were introduced in Section 1.3.

Any number on a number line, including fractions and decimals and any number in between them, is called a **real** number. The set of 'real' numbers is denoted by $\mathbb{R}$. Unless otherwise stated, you can usually assume that a domain is the real number set, or a part of it. Other commonly used sets of numbers are the **integers**, which are the positive and negative 'whole' numbers

$$\ldots, -2, -1, 0, 1, 2, \ldots$$

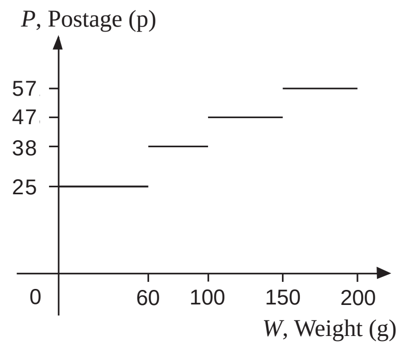
Other domains and ranges are usually described using set notation, like the range in this next example.

Example

The table below shows the current postal rates in the U.K. for first class letters.

Weight up to	60 g	100 g	150 g	200 g
Postage	25p	38p	47p	57p

This is a mapping from part of the real numbers (between 0 and 200), to a 'discrete' range. This means that the range contains only certain values. In this case, the range is the set of numbers {25, 38, 47, 57}. The graph of this mapping is shown opposite.



A vertical line drawn across the graph will cut it only once, so this mapping is a function, even though it does not appear to have a 'formula' of the usual sort. It is possible to write down a kind of formula, however ;

$$\text{Postage, } P = \begin{cases} 25 \text{ for } 0 < W \le 60 \\ 38 \text{ for } 60 < W < 100 \\ \ldots\ldots\ldots\ldots \text{etc.} \end{cases}$$

The value of P changes in steps as the weight increases, so it is necessary to have different rules for different parts of the domain.

Activity 4

Use a graph plotting program or a graphic calculator to make sketches illustrating the following rules. By looking at the sketches, decide on a domain which will make the rule a function.

(a) $y = \frac{1}{x-5}$ (b) $y = \frac{3-x}{1-x}$ (c) $y = +\sqrt{x-2}$

(d) $y = 1 - x^2$ (e) $y = \frac{1}{x^2}$ (f) $y = \frac{2}{1-x^3}$.

Function notation

Returning to the mapping of volume to pressure met earlier in this section, another way of writing the mapping or function is illustrated by

$$p: v \mapsto \frac{k}{v},\ v \in \mathbb{R},\ v > 0.$$

This is read as p is the function which maps v to $\frac{k}{v}$ where v is any real number greater than zero. The more usual way of writing a function is

$$p(v) = \frac{k}{v},\ v \in \mathbb{R},\ v > 0$$

which illustrates that p is a function of v. In what follows the second method will generally be used, but you should be aware of the alternative.

Another example is given by the function

$$F: t \mapsto +\sqrt{t},\ t \in \mathbb{R},\ t \geq 0$$

or in the usual notation

$$F(t) = +\sqrt{t},\ t \in \mathbb{R},\ t \geq 0\,.$$

Hence, when $t = 4,\ F(4) = +\sqrt{4} = 2,$ and in general if $t = a,$

$$F(a) = +\sqrt{a};$$

when $t = b^2,$

$$F\left(b^2\right) = +\sqrt{b^2} = b.$$

Exercise 2A

1. f is a function defined by the equation
   $f(x) = x^2 + 2.$
   Find the value of
   (a) $f(2)$ (b) $f(-1)$ (c) $f(0)$
   (d) $f(a^2)$ (e) $f(1-a)$
   (where a is a constant real number).

2. g is defined by $g(x) = \frac{1}{x}$
   Find the value of the following if possible
   (a) $g(1)$ (b) $g(-1)$ (c) $g(0)$
   (d) $g(a^2)$ (e) $g(1-a)$
   (where a is a constant real number).

3. For each of the following rules, use a graphic calculator or computer to make a sketch, for values of x between -10 and $+10$. Use your sketches to work out a domain for each mapping which will make each one a function.
   (a) $f : x \mapsto \frac{1}{x^3}$ (b) $g : x \mapsto x^5$
   (c) $h : x \mapsto \frac{1}{x+2}$ (d) $m : x \mapsto +\sqrt{1-x}$.

4. The wind chill temperature, T°C, depends on the actual temperature, t°C, and the wind speed, v mph; an appropriate formula for T is given by
   $T = 33 + \left(0.45 + 0.29\sqrt{v} - 0.02v\right)(t - 33)$
   for $t > -273°$, $v \geq 5$. Sketch a graph of T against t for varying wind speeds; for example $v = 10$, 15 and 20 mph.

2.2 Some important graphs

Many of the functions which arise from problems can be 'built up' from simpler functions. In this section you will see how some of these simpler functions behave by looking at their graphs. This will help you to sketch more complicated functions later on.

Activity 5 Some well known curves

(a) Use a computer or calculator to make sketches of the following curves on the same pair of axes. Make the sketches for values of x between -2 and $+2$, and the y values between -20 and $+20$.

$$y = x,\ y = x^2,\ y = x^3,\ y = x^4,\ y = x^5,\ y = x^6.$$

Which points do all the curves pass through? What happens to the curves between $x = 0$ and $x = 1$ as the power of x increases? What happens as the power increases for values of x greater than 1? Try to predict what the curves $y = x^7$ and $y = x^8$ will look like and check your answer on the computer or calculator.

(b) Sketch these curves on separate axes with x-axis from -2 to 2 and y from -10 to 10.

$$y = x^2,\ y = x^4,\ y = \frac{1}{x^2},\ y = \frac{1}{x^4}$$

Describe any symmetry these graphs have.

(c) Now sketch these graphs on four pairs of axes like the ones you have just used.

$$y = x,\ y = x^3,\ y = \frac{1}{x},\ y = \frac{1}{x^3}$$

Describe any symmetry these graphs have. (Use the term 'rotational symmetry' in the description).

Activity 6 Odd and even powers

The first four graphs in part (b) of Activity 5 were all for even powers of x, whilst the graphs in part (c) were all for odd powers of x. Use the knowledge you have gained from these graphs to describe the symmetry of the following graphs. Then check your answers by using a computer or calculator to see the graphs.

(a) $y = x^{10}$, (b) $y = x^{11}$, (c) $y = \frac{1}{x^5}$,

(d) $y = x^2 - 3$, (e) $y = x^3 + 1$, (f) $y = x - 2$.

Were you surprised by any of the graphs? If so, try to find out why you were wrong.

Activity 7 Fractional powers

It is possible to find the value of a fractional power. Chapter 9 covers this in more detail. Using your graph plotting device, sketch these curves on the same axes, with x values from -1 to $+2$ and y values from -1 to 8.

$y = x^{\frac{1}{2}}$, $y = x^{\frac{1}{3}}$, $y = x^1$, $y = x^{\frac{3}{2}}$,

$y = x^2$, $y = x^{\frac{5}{2}}$, $y = x^3$.

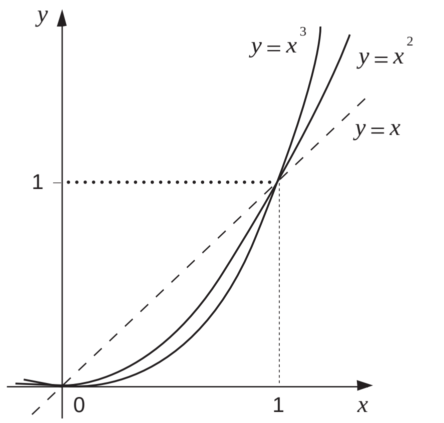
Which points do all the curves have in common?

What happens as the power of x increases when x is between 0 and 1, and when x is more than 1?

Do all the curves exist for x values less than zero?

Shapes of graphs

The graph for any power of x will pass through the points (0, 0) and (1, 1). When x is between 0 and 1, the higher the power of x, the lower the result becomes. For example



Power of $\frac{1}{2}$	$\left(\frac{1}{2}\right)^0$	$\left(\frac{1}{2}\right)^1$	$\left(\frac{1}{2}\right)^2$	$\left(\frac{1}{2}\right)^3$	$\left(\frac{1}{2}\right)^4$
Result	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$

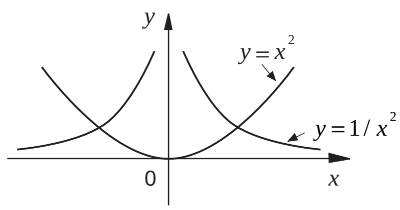
This means that the graphs of powers of x get 'flatter' as the power increases when x is between 0 and 1. This is shown in the figure opposite.

When x is larger than 1, the higher the power of x the larger the result:

Power of 2	2^0	2^1	2^2	2^3	2^4
Result	1	2	4	8	16

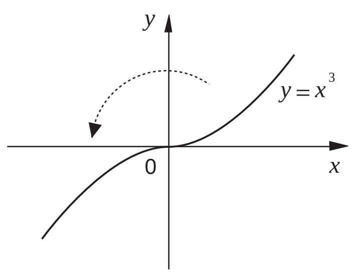
This means that the curve of a higher power of x will be higher than the curve of a lower power when x is greater than 1. So the curve of a higher power 'overtakes' the curve of a lower power when $x = 1$. This is illustrated above.

*Odd and even functions



The curves for even powers of x are symmetrical about the y-axis. Any function whose curve has the y-axis as a line of symmetry is therefore called an **even** function. The graph of two even functions are shown opposite.

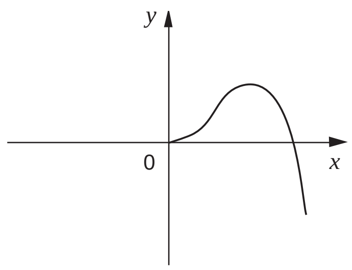
Curves for odd powers of x have two fold rotational symmetry about (0, 0). That is, the right hand side of the curve can be rotated through 180° about (0, 0) so that it fits onto the left hand side. This is illustrated for the graph of $y = x^3$. Functions having graphs with this kind of rotational symmetry are called **odd** functions.



Obviously, if you know already that a function is even or odd, you can easily sketch the whole of its graph if you know one half of it.

Example

One part of the graph of $y = f(x)$ is shown opposite. Complete the curve assuming that

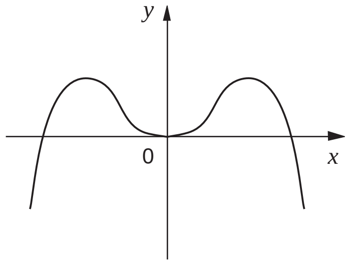


(a) $f(x)$ is even,

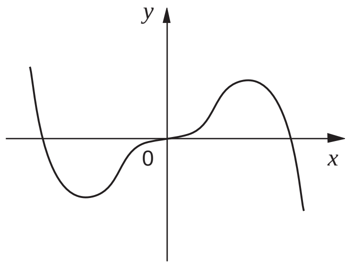
(b) $f(x)$ is odd.

Solution

(a) If $f(x)$ is even, the curve must be symmetrical about the y-axis, so the curve will be the one shown opposite.



(b) If $f(x)$ is odd, the curve has two fold rotational symmetry about (0, 0). This produces the graph below.



Most functions are neither even nor odd. For a function to be even, its equation must only contain even powers of x. For a graph to be odd, only odd powers of x may appear. (There are some functions which are not usually given in terms of powers of x, like the trigonometrical functions, which are even or odd however). A mixture of odd and even powers of x means the graph is neither odd nor even.

Example

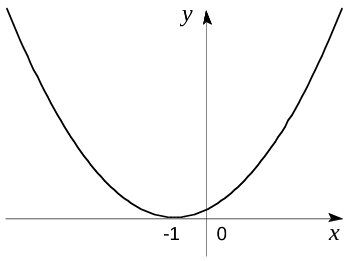
Is $y = (x+1)^2$ even?

Solution

When the brackets are multiplied out the reason the function is **not** even is clear :

$$\begin{aligned} y &= (x+1)^2 \\ &= (x+1)(x+1) \\ &= x(x+1)+1(x+1) \\ &= x^2+x+x+1 \\ &= x^2+2x+1. \end{aligned}$$

This equation contains an even power (x^2) and an odd power (x), so the graph does not have reflection symmetry about the y-axis. The graph is sketched opposite. It is symmetrical, but about the line $x = -1$, not the y-axis.



An alternative way of defining odd and even functions is to say that

$$f(x) \text{ is even if } f(-x) = f(x)$$
$$f(x) \text{ is odd if } f(-x) = -f(x).$$

Example

Are the following functions odd, even or neither?

(a) $f(x) = x^2+1$ (b) $f(x) = x^3$ (c) $f(x) = \dfrac{1}{x+1}$, $(x \neq -1)$.

Solution

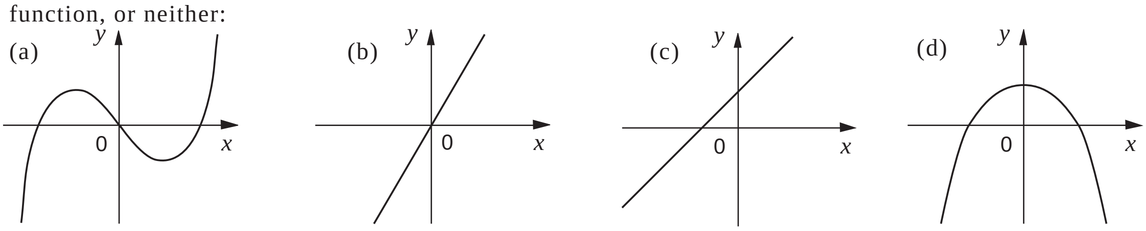
(a) $f(-x) = (-x)^2 + 1 = x^2 + 1 = f(x)$ – hence even.

(b) $f(-x) = (-x)^3 = -x^3 = -f(x)$ – hence odd.

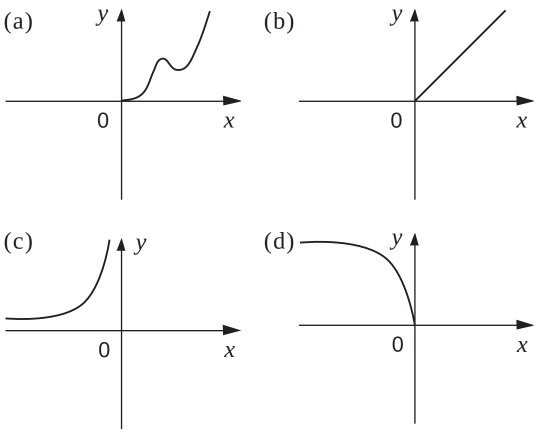
(c) $f(-x) = \dfrac{1}{-x+1} \neq \pm f(x)$ – hence neither even or odd.

*Exercise 2B

1. Decide whether the graphs illustrate an even function, an odd function, or neither:



2. Copy and complete the following curves, assuming that the function each represents is i) even, and ii) odd.



3. By looking only at the equation decide whether these functions are even, odd or neither.

   (a) $y = 3x$ (b) $y = 3x + 1$

   (c) $y = (x+2)^2$ (d) $y = x^2 + 2$

   (e) $y = \left(x^2 + 2\right)^2$

   (You will need to multiply out any bracket. Also note that a constant number can be thought of as an even power of x.
   For instance $2x^0 = 2 \times 1 = 2$ as $x^0 = 1$.)

2.3 Miscellaneous Exercises

1. Many shoes give both their U.K. size and European equivalent. For example,

   English adult size 12 = European size 47

   English adult size 5 = European size 38

   Use this information to construct a conversion graph between the two sizes. What does English size 0 correspond to in terms of European size?

   (English adult size 0 is in fact equivalent to junior size 13).

2. Give a reason why the domains for each of these functions are unsuitable, and give a domain that is acceptable. Also state the range.

   (a) $f(x) = \dfrac{1}{\sqrt{x}}$, $x \in \mathbb{R}$

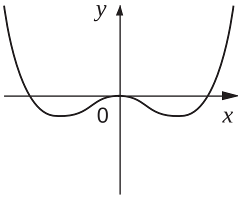
   (b) $f(x) = \dfrac{1}{x-3}$, $x > 0$

   (c) $f(x) = \sqrt{6-x}$, $x > 0$

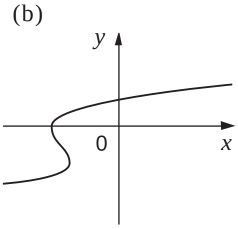
   (d) $f(x) = \dfrac{1}{(x-2)(x+3)}$, $x \in \mathbb{R}$.

3. Which of the graphs below represent functions?

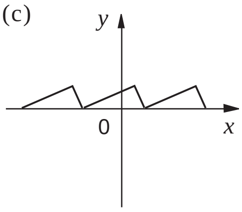
(a)



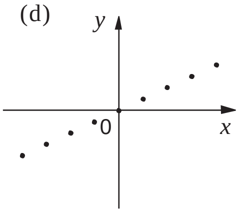
(b)



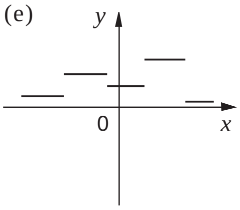
(c)



(d)



(e)

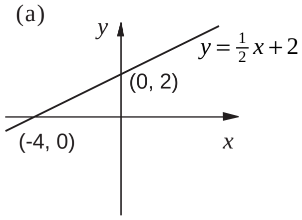


4. Let $f: x \mapsto \dfrac{1}{x^3 - 8}$, $x \in \mathbb{R}$, $x \neq 2$. Find:

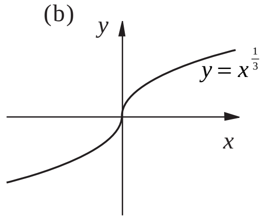
(a) $f(0)$ (b) $f(1)$ (c) $f(-1)$ (d) $f(-2)$.

*5. State whether each of the following graphs represents an even or an odd function, or neither.

(a)



(b)



(c)