1.8

COMBINATIONS OF FUNCTIONS: COMPOSITE FUNCTIONS

What You Should Learn

- Add, subtract, multiply, and divide functions.
- Find the composition of one function with another function.
- Use combinations and compositions of functions to model and solve real-life problems.

Arithmetic Combinations of Functions

Arithmetic Combinations of Functions

Given: $f(x) = 2x - 3$ and $g(x) = x^2 - 1$

$f(x) + g(x) = (2x - 3) + (x^2 - 1) = x^2 + 2x - 4$ Sum

$f(x) - g(x) = (2x - 3) - (x^2 - 1) = -x^2 + 2x - 2$ Difference

$f(x)g(x) = (2x - 3)(x^2 - 1) = 2x^3 - 3x^2 - 2x + 3$ Product

$$\frac{f(x)}{g(x)} = \frac{2x - 3}{x^2 - 1}, \quad x \neq \pm 1$$ Quotient

Arithmetic Combinations of Functions

The domain of an **arithmetic combination** of functions f and g consists of all real numbers that are **common** to the domains of f and g.

In the case of the quotient $f(x)/g(x)$, there is the further restriction that $g(x) \neq 0$.

Arithmetic Combinations of Functions

Sum, Difference, Product, and Quotient of Functions

Let f and g be two functions with overlapping domains. Then, for all x common to both domains, the *sum*, *difference*, *product*, and *quotient* of f and g are defined as follows.

1. *Sum:* $(f + g)(x) = f(x) + g(x)$

2. *Difference:* $(f - g)(x) = f(x) - g(x)$

3. *Product:* $(fg)(x) = f(x) \cdot g(x)$

4. *Quotient:* $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}, \quad g(x) \neq 0$

Example 1 – *Finding the Sum of Two Functions*

Given $f(x) = 2x + 1$ and $g(x) = x^2 + 2x – 1$ find $(f + g)(x)$. Then evaluate the sum when $x = 3$.

Solution:

$(f + g)(x) = f(x) + g(x) = (2x + 1) + (x^2 + 2x – 1)$

$= x^2 + 4x$

When $x = 3$, the value of this sum is

$(f + g)(3) = 3^2 + 4(3)$

$= 21.$

Composition of Functions

Composition of Functions

Given $f(x) = x^2$ and $g(x) = x + 1$, the composition of f with g is

$$f(g(x)) = f(x + 1)$$

$$= (x + 1)^2.$$

This composition is denoted as $f \circ g$

and reads as "f composed with g."

Composition of Functions

Definition of Composition of Two Functions

The **composition** of the function f with the function g is

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f. (See Figure 1.90.)

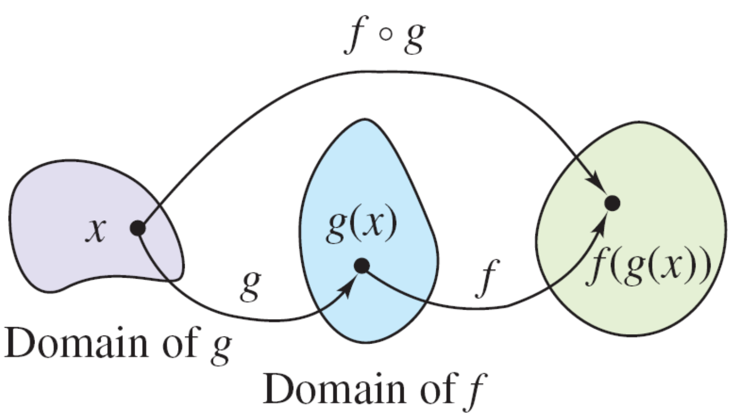

Figure 1.90

Example 5 – *Composition of Functions*

Given $f(x) = x + 2$ and $g(x) = 4 - x^2$, find the following.

a. $(f \circ g)(x)$ **b.** $(g \circ f)(x)$ **c.** $(g \circ f)(-2)$

Solution:

a. The composition of *f* with *g* is as follows.

$(f \circ g)(x) = f(g(x))$ Definition of $f \circ g$

$= f(4 - x^2)$ Definition of $g(x)$

$= (4 - x^2) + 2$ Definition of $f(x)$

$= -x^2 + 6$ Simplify.

Example 5 – *Solution*

cont'd

b. The composition of g with f is as follows.

$$(g \circ f)(x) = g(f(x)) \qquad \text{Definition of } g \circ f$$

$$= g(x + 2) \qquad \text{Definition of } f(x)$$

$$= 4 - (x + 2)^2 \qquad \text{Definition of } g(x)$$

$$= 4 - (x^2 + 4x + 4) \qquad \text{Expand.}$$

$$= -x^2 - 4x \qquad \text{Simplify.}$$

Note that, in this case, $(f \circ g)(x) \neq (g \circ f)(x)$.

Example 5 – *Solution*

cont'd

c. Using the result of part (b), you can write the following.

$$(g \circ f)(-2) = -(-2)^2 - 4(-2)$$ Substitute.

$$= -4 + 8$$ Simplify.

$$= 4$$ Simplify.

Composition of Functions

Given $h(x) = (3x - 5)^3$ is the composition of f with g where $f(x) = x^3$ and $g(x) = 3x - 5$.

$$h(x) = (3x - 5)^3 = [g(x)]^3 = f(g(x)).$$

Application

Example 8 – *Bacteria Count*

The number N of bacteria in a refrigerated food is given by

$N(T) = 20T^2 - 80T + 500, \qquad 2 \le T \le 14$

where T is the temperature of the food in degrees Celsius.

When the food is removed from refrigeration, the temperature of the food is given by

$T(t) = 4t + 2, \; 0 \le t \le 3$

where t is the time in hours.

(a) Find the composition $N(T(t))$ and interpret its meaning in context.

(b) Find the time when the bacteria count reaches 2000.

Example 8(a) – *Solution*

$N(T(t)) = 20(4t + 2)^2 - 80(4t + 2) + 500$

$= 20(16t^2 + 16t + 4) - 320t - 160 + 500$

$= 320t^2 + 320t + 80 - 320t - 160 + 500$

$= 320t^2 + 420$

The composite function $N(T(t))$ represents the number of bacteria in the food as a function of the amount of time the food has been out of refrigeration.

Example 8(b) – *Solution*

cont'd

The bacteria count will reach 2000 when $320t^2 + 420 = 2000$. Solve this equation to find that the count will reach 2000 when $t \approx 2.2$ hours.

When you solve this equation, note that the negative value is rejected because it is not in the domain of the composite function.