FUNCTIONS - PART 2

Introduction

This handout is a summary of the basic concepts you should understand and be comfortable working with for the second math review module on functions. This is intended as a summary and should be used together with the references given below.

References: Chapter 2 in *Precalculus* by Stewart, Redlin and Watson. Chapter 5 in the *Mathematics Review Manual* by Lovric available online at www.math.mcmaster.ca/lovric/rm.html

If you are not familiar with any of the material below you need to spend time studying these concepts and doing some exercises.

Symmetry of Functions

Even and Odd Functions

A function f is even if for all x in the domain of f we have $f(-x) = f(x)$. The graph of an even function is symmetric with respect to the y-axis. For example, $f(x) = x^2 - 3$ is an even function.

A function f is odd if for all x in the domain of f $f(-x) = -f(x)$. The graph of an odd function is symmetric with respect to the origin, i.e., if (x, y) is a point on the graph, then so is $(-x, -y)$. For example, $f(x) = x^3$ is an odd function.

Example: Determine whether or not $f(x) = x^5 + x$ is even, odd or neither.

Note that even and odd functions are both defined by equations which have $f(-x)$ on the left side of the equality sign. Therefore, to show if a function is even, odd or neither we start with $f(-x)$ and then try to see if this is equal to right side of the equality sign in the definition of even and odd functions. For our example we have,

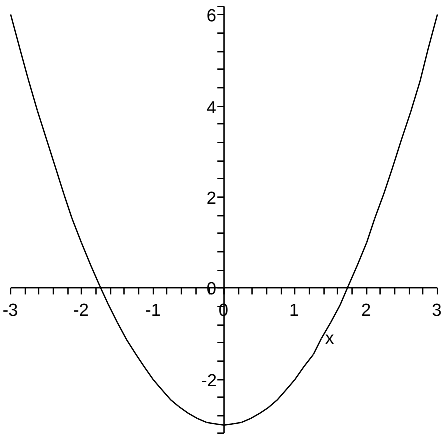

Figure 1: Graph of the even function $f(x) = x^2 - 3$.

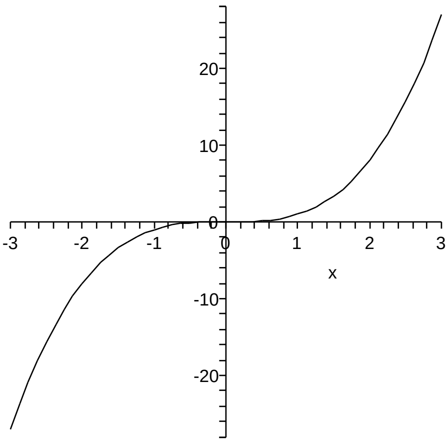

Figure 2: Graph of the odd function $f(x) = x^3$.

$$
\begin{aligned}
f(-x) &= (-x)^5 + (-x) \\
&= (-1)^5 x^5 - x \\
&= -x^5 - x \\
&= -(x^5 = x) \\
&= -f(x)
\end{aligned}
$$

Therefore, we have that $f(-x) = -f(x)$ and this is the requirement for a function to be odd. So, $f(x) = x^5 + x$ is an odd function.

Exercise: Is $f(x) = 1 - x^4$ even, odd or neither?

Answer: Even.

Exercise: Is $f(x) = 2x - x^2$ even, odd or neither?

Answer: Neither.

Composition of Functions

Given two functions f and g, the composite function $f \circ g$ is defined by

$$(f \circ g)(x) = f(g(x))$$

Let A be the domain of f and B be the domain of g. Then, the domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f. In set notation,

$$\text{domain of } f \circ g = \{x \in A \,|\, g(x) \in B\}$$

Example: Let $f(x) = x^2$ and $g(x) = \sqrt{x-3}$. Find the functions $f \circ g$ and $g \circ f$ and their domains.

From the definition of the composition of functions we have:

$$(f \circ g)(x) = f(g(x)) = f(\sqrt{x-3}) = (\sqrt{x-3})^2 = x - 3$$

The domain of $f \circ g$ is the set of x in the domain of g such that $g(x)$ is in the domain of f. That is, the domain of $f \circ g$ is the domain of g except that we have the added restriction that $g(x)$ must be in the domain of f. So, the way we determine the domain of $f \circ g$ is to first determine the domain of g and then see what additional restrictions are imposed by the fact that $g(x)$ must lie in the domain of f. The domain of g is:

$$\{x \mid x - 3 \geq 0\} = \{x \mid x \geq 3\} = [3, \infty)$$

Then, the domain of f is $\mathbb{R}$. Since $g(x) \in \mathbb{R}$ for all x in the domain of g there are no additional restrictions imposed. Therefore, the domain of $f \circ g$ is $[3, \infty)$.

Now, for $g \circ f$ we have:

$$(g \circ f)(x) = g(f(x)) = g(x^2) = \sqrt{x^2 - 3}$$

The domain of $g \circ f$ is the set of x in the domain of f such that $f(x)$ is in the domain of g. The domain of f is $\mathbb{R}$. Since the domain of g is $[3, \infty)$ we need to determine for what values of $x \in \mathbb{R}$ is $f(x) \in [3, \infty)$. Since $f(x) = x^2$, then $x^2 \in [3, \infty)$ if and only if $x \in (-\infty, -\sqrt{3}] \cup [\sqrt{3}, \infty)$. Therefore, the domain of $g \circ f$ is $(-\infty, -\sqrt{3}] \cup [\sqrt{3}, \infty)$.

<u>Exercise</u>: Given the functions f and g as defined above, find the composite functions $f \circ f$ and $g \circ g$ and find their domains.

Answer: $(f \circ f)(x) = x^4$ and its domain is $\mathbb{R}$. $(g \circ g)(x) = \sqrt{\sqrt{x - 3} - 3}$ and its domain is $[12, \infty)$.

Composition of More Than Two Functions

We can also apply the above idea of composition of functions to three or more functions. For example, given three functions f, g and h the composition $f \circ g \circ h$ is defined as $(f \circ g \circ h)(x) = f(g(h(x)))$.

Example: Let $f(x) = \dfrac{1}{x}$, $g(x) = x^3$ and $h(x) = x^2 + 2$. Find $f \circ g \circ h$.

$$\begin{aligned}
(f \circ g \circ h)(x) &= f(g(h(x))) \\
&= f(g(x^2 + 2)) \\
&= f((x^2 + 2)^2) \\
&= \frac{1}{(x^2 + 3)^2}
\end{aligned}$$

One-to-One Functions

A function f with domain A is called a one-to-one function if for any $x_1, x_2 \in A$ with $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

To prove that a function is one-to-one we suppose that there are x_1 and x_2 in the domain of the function such that $f(x_1) = f(x_2)$. Then we use this equation to show that in fact we must have that $x_1 = x_2$.

Example: Let f(x)=7x+2. Prove that f is one-to-one.

As stated above suppose that we have x_1 and x_2 such that $f(x_1) = f(x_2)$. Then we have

$$\begin{aligned} f(x_1) &= f(x_2) \\ 7x_1 + 2 &= 7x_2 + 2 \\ 7x_1 &= 7x_2 \\ x_1 &= x_2 \end{aligned}$$

We have shown that if $f(x_1) = f(x_2)$ then, in fact, $x_1 = x_2$. Therefore, we can conclude that f is a one-to-one function.

<u>Exercise</u>: Let $f(x) = \dfrac{7}{2x-1}$. Prove that f is one-to-one.

To show that a function is not one-to-one we just need to find an example of x_1 and x_2 with $x_1 \neq x_2$ in the domain of the function such that $f(x_1) = f(x_2)$.

Example: Let $f(x) = x^2$. Prove that f is not one-to-one.

Let $x_1 = 2$ and $x_2 = -2$. So we have $x_1 \neq x_2$. Then $f(x_1) = f(2) = 2^2 = 4$ and $f(x_2) = f(-2) = (-2)^2 = 4$. So, $f(x_1) = f(x_2)$. This shows that $f(x) = x^2$ is not one-to-one.

Inverse Functions

Let f be a one-to-one function with domain A and range B. Then its inverse function f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \Leftrightarrow f(x) = y$$

Finding f^{-1} for Specific Values

Suppose for a given one-to-one function f, $f(1) = 5$, $f(3) = 7$ and $f(8) = -10$. Find $f^{-1}(5)$, $f^{-1}(7)$ and $f^{-1}(-10)$.

Using the definition of the inverse function we can conclude that since $f(1) = 5$ then $f^{-1}(5) = 1$. Since $f(3) = 7$ then $f^{-1}(7) = 3$. Since $f(8) = -10$ then $f^{-1}(-10) = 8$.

Property of Inverse Functions

Let f be a one-to-one function with domain A and range B. The inverse function f^{-1} satisfies the following cancelation properties.

$$f^{-1}(f(x)) = x \quad \text{for all } x \text{ in } A$$

$$f(f^{-1}(x)) = x \quad \text{for all } x \text{ in } B$$

Conversely, any function f^{-1} satisfying these equations is the inverse of f.

These properties indicate that f is the inverse function of f^{-1}. So, we can say that f and f^{-1} are inverses of each other.

Verifying that Two Functions are Inverses

Example: Show that $f(x) = x^3$ and $g(x) = x^{1/3}$ are inverses of each other.

To show that these functions are inverses of each other we use the property on inverse functions. Note that the domain and range of both f and g is $\mathbb{R}$. We have

$$g(f(x)) = g(x^3) = (x^3)^{1/3} = x$$

$$f(g(x)) = f(x^{1/3}) = (x^{1/3})^3 = x$$

So, by the property of inverse functions, f and g are inverses of each other.

<u>Exercise</u>: Show that $f(x) = 3x + 4$ and $g(x) = \dfrac{4 - x}{3}$ are inverses of each other.

How to Find the Inverse of a One-to-One Function

Given a one-to-one function f we can find its inverse by applying the following procedure.

1. Write $y = f(x)$.
2. Solve this equation for x in terms of y (if possible).
3. Interchange x and y. The resulting equation is $y = f^{-1}(x)$.

Example: Find the inverse function of $f(x) = (2 - x^3)^5$.

We apply the above three step procedure. We write $y = (2 - x3)^5$. Then, we solve this equation for x.

$$\begin{aligned}
y &= (2 - x^3)^5 \\
y^{1/5} &= 2 - x^3 \\
x^3 &= 2 - y^{1/5} \\
x &= (2 - y^{1/5})^{1/3} \\
y &= (2 - x^{1/5})^{1/3} \quad \text{[Interchange } x \text{ and } y \text{ (Step 3)]}
\end{aligned}$$

Therefore, the inverse function is $f^{-1} = (2 - x^{1/5})^{1/3}$.

We can check that this is correct by using the property of inverses.

$$\begin{aligned}
f(f^{-1}(x)) &= f((2 - x^{1/5})^{1/3}) \\
&= (2 - ((2 - x^{1/5})^{1/3})^3)^5 \\
&= (2 - (2 - x^{1/5}))^5 \\
&= (x^{1/5})^5 \\
&= x
\end{aligned}$$

$$\begin{aligned}
f^{-1}(f(x)) &= f^{-1}((2 - x^3)^5) \\
&= (2 - ((2 - x^3)^5)^{1/5})^{1/3} \\
&= (2 - (2 - x^3))^{1/3} \\
&= (x^3)^{1/3} \\
&= x
\end{aligned}$$

Therefore, we have that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$, so the function we found is indeed the inverse of f.

Exercise: Find the inverse function of $f(x) = \dfrac{6}{5-x}$. What is the domain and range of the inverse function.

Answer: The inverse function is: $f^{-1}(x) = \dfrac{5x-6}{x}$. The domain is $(-\infty, 0) \cup (0, \infty)$. The range is $(-\infty, 5) \cup (5, \infty)$.